3.7 Implicit Differentiation
Day 2

1. Differentiate both sides with respect to x.

2. Get all terms with dy/dx to one side of the equation.

3. Factor out dy/dx.

4. Solve for dy/dx

Ex 1) Find the slope of the tangent

to $y^2 - x^2 = 1$ at $(1, \sqrt{2})$

$$2y \cdot \frac{dy}{dx} - 2x = 0$$

$$2y \cdot \frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{\cancel{2}x}{\cancel{2}y} = \boxed{\frac{x}{y}}$$

$\underline{\text{Find slope}}$ $(1, \sqrt{2})$

$$\frac{dy}{dx} = \frac{1}{\sqrt{2}} = \boxed{\frac{\sqrt{2}}{2}}$$

$vu'+uv'$

Ex 2) $x^2 - \overset{uv}{xy} + y^2 = 1$

Find $\frac{dy}{dx}$

Find $\frac{d^2y}{dx^2}$ Do on next slide

$$2x - \left(y\cdot 1 + x\cdot 1\cdot\frac{dy}{dx}\right) + 2y\cdot\frac{dy}{dx} = 0$$

$$2x - y - x\frac{dy}{dx} + 2y\cdot\frac{dy}{dx} = 0$$

$$-x\frac{dy}{dx} + 2y\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx}(-x+2y) = y - 2x$$

$$\frac{dy}{dx} = \boxed{\frac{y-2x}{2y-x}}$$

$\frac{dy}{dx} = y' = \frac{\overset{u}{y-2x}}{\underset{v}{2y-x}}$ $\boxed{\frac{y-2x}{2y-x}} = \frac{dy}{dx}$

Now find $\frac{d^2y}{dx^2} = y''$

$$y'' = \frac{(2y-x)\left(1\cdot\frac{dy}{dx} - 2\right) - (y-2x)\left(2\cdot\frac{dy}{dx} - 1\right)}{(2y-x)^2}$$

$$y'' = \frac{(2y-x)\left(\frac{y-2x}{2y-x} - 2\right) - (y-2x)\left(2\cdot\frac{y-2x}{2y-x} - 1\right)}{(2y-x)^2}$$

Ex 3) $y^2 + 2x - 4y - 1 = 0$ Find the tangent and normal line at $(\overset{}{\underset{x}{-2}}, \underset{y}{1})$.

$$2y\cdot\frac{dy}{dx} + 2 - 4\cdot\frac{dy}{dx} = 0$$

$$2y\cdot\frac{dy}{dx} - 4\frac{dy}{dx} = -2$$

$$\frac{dy}{dx}(2y-4) = -2$$

$$\frac{dy}{dx} = \frac{-2}{2y-4} = \frac{-1}{y-2} = \frac{-1}{1-2} = \frac{-1}{-1} = 1$$

Tangent

$$y - 1 = 1(x+2)$$

Normal

$$y - 1 = -1(x+2)$$

Ex 4) $\overset{u}{x} \overset{\cdot}{} \sin \overset{v}{2y} = \overset{u}{y} \overset{\cdot}{} \cos \overset{v}{2x}$ $vu' + uv'$

Find the tangent and normal line at (π/4, π/2)

$$\sin(2y)\cdot 1 + x\cdot\cos(2y)\cdot 2\cdot\frac{dy}{dx} = \cos(2x)\cdot 1\cdot\frac{dy}{dx} + y\cdot -\sin(2x)\cdot 2$$

$$2x\cos(2y)\frac{dy}{dx} - \cos(2x)\frac{dy}{dx} = -\sin(2y) - 2y\sin(2x)$$

$$\frac{dy}{dx}(2x\cos(2y) - \cos(2x) = -\sin(2y) - 2y\sin(2x)$$

$$\frac{dy}{dx} = \frac{-\sin(2y) - 2y\sin(2x)}{2x\cos(2y) - \cos(2x)}$$

$$\oplus \quad = \frac{-\sin(\cancel{2}\cdot\frac{\pi}{\cancel{2}}) - \cancel{2}\cdot\frac{\pi}{\cancel{2}}\cdot\sin(\cancel{2}\cdot\frac{\pi}{\underset{2}{\cancel{4}}})}{\cancel{2}(\frac{\pi}{\underset{2}{\cancel{4}}})\cos(\cancel{2}\cdot\frac{\pi}{\cancel{2}}) - \cos(\cancel{2}\cdot\frac{\pi}{\underset{2}{\cancel{4}}})}$$

$$= \frac{0 - \pi\cdot 1}{\frac{\pi}{2}\cdot -1 - 0} = \frac{-\pi}{-\pi/2} = -\pi\cdot\frac{2}{-\pi} = \boxed{2}$$

$(\pi/4, \pi/2)$

Tangent

$$y - \pi/2 = 2(x - \pi/4)$$

Normal

$$y - \pi/2 = -\frac{1}{2}(x - \pi/4)$$

Ex 5) $\overset{u}{x^2}\cdot\overset{v}{\cos^2 y} - \sin y = 0$ $(uv)' = vu' + uv'$

Find the tangent and normal line at (0, π)

$$\cos^2 y\cdot 2x + x^2\cdot 2\cos y\cdot -\sin y\cdot\frac{dy}{dx} - \cos y\cdot\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(-2x^2\cos y\sin y - \cos y\right) = -2x\cos^2 y$$

$$\frac{dy}{dx} = \frac{-2x\cos^2 y}{-2x^2\cos y\sin y - \cos y} = \frac{-2(0)(\cos^2\pi)}{-2(0)\cos\pi\sin\pi - \cos\pi}$$

$$= \frac{0}{0+1} = \textcircled{0} \quad \frac{0}{1}$$

Tangent $m=0$ $(0,\pi)$

$y-\pi = 0(x-0)$

$\boxed{y=\pi}$

Normal: $m=$ undefined

$\boxed{x=0}$ $(0,\overset{0}{\pi})$