6.4 Exponential Growth and Decay

Ex 1) Solve the initial value problem.

$$\frac{dy}{dx} = -\frac{x}{y}$$

$x = 4, y = 3$

$y \cdot dy = -x\,dx$

$\int y\,dy = \int -x\,dx$

$\frac{y^2}{2} = \frac{-x^2}{2} + C$

$\frac{3^2}{2} = \frac{-4^2}{2} + C$

$\frac{9}{2} = \frac{-16}{2} + C$

$\frac{25}{2} = C$

$\frac{y^2}{2} = \frac{-x^2}{2} + \frac{25}{2}$

$\sqrt{y^2} = \sqrt{-x^2+25}$

$\boxed{y = \sqrt{25-x^2}}$

Steps

1. Separate the variables
2. Integrate/Antideriivative
3. Solve for c.
4. Solve for y.

Ex 2) Solve the initial value problem

$$\frac{dy}{dx} = 2xy \qquad x = 0,\ y = 3$$

$\ln x = y$

$e^y = x$

$\frac{1}{y}dy = 2x\,dx$

$\int \frac{1}{y}dy = \int 2x\,dx$

$\ln y = x^2 + C$

$\ln 3 = 0^2 + C$

$\ln 3 = C$

$\ln y = \underbrace{x^2 + \ln 3}$

$\boxed{e^{x^2+\ln 3} = y}$

OR $\ln y - \ln 3 = x^2$

$\ln \frac{y}{3} = x^2$

$e^{x^2} = \frac{y}{3}$

$\boxed{3e^{x^2} = y}$

Ex3) Solve the initial value problem.

$$\frac{dy}{dx} = \cos^2 y \qquad x = 0, y = 0$$

$$\frac{1}{\cos^2 y}\,dy = dx$$

$$\sec^2 y\,dy = dx$$

$$\int \sec^2 y\,dy = \int 1\,dx$$

$$\tan y = x + C$$

$$\tan 0 = 0 + C$$

$$0 = C$$

$$\tan y = x$$

$$\boxed{\tan^{-1}(x) = y}$$

Ex 4) Solve the initial value problem $x^2 \cdot x^m = x^{2+m}$

$$\frac{dy}{dx} = e^{x-y}$$

$x = 0, y = 2$

$$\frac{dy}{dx} = e^x \cdot e^{-y}$$

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$

$$e^y \cdot dy = e^x \cdot dx$$

$$\int e^y \cdot dy = \int e^x \, dx$$

$$e^y = e^x + C$$

$$e^2 = e^0 + C$$

$$e^2 = 1 + C$$

$$e^2 - 1 = C$$

$$e^y = \underbrace{e^x + e^2 - 1}$$

$$\boxed{\ln(e^x + e^2 - 1) = y}$$

$$\boxed{\begin{array}{l} e^y = x \\ \ln x = y \end{array}}$$

Ex 5) Solve the initial value problem

$$\frac{dy}{dx} = \frac{4\sqrt{(y)}\ \ln x}{x} \qquad x = e,\ y = 1$$

$$\frac{1}{4\sqrt{y}}dy = \frac{\ln x}{x}\cdot dx$$

$$\frac{1}{4}\int y^{-1/2}\cdot dy = \int \frac{\ln x}{x}\cdot dx \qquad \text{let } u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{1}{4}\cdot y^{1/2}\cdot \frac{2}{1} = \int u\cdot du \qquad du = \frac{1}{x}\cdot dx$$

$$\frac{1}{2}\sqrt{y} = \frac{u^2}{2} + C$$

$$\frac{1}{2}\sqrt{y} = \frac{\ln^2 x}{2} + C$$

$$\frac{1}{2}\sqrt{1} = \frac{(\ln e)^2}{2} + C$$

$$\frac{1}{2} = \frac{1}{2} + C \qquad \underline{\underline{C=0}}$$

$$\frac{1}{2}\sqrt{y} = \frac{\ln^2 x}{2}$$

$$(\sqrt{y})^2 = (\ln^2 x)^2$$

$$\boxed{y = \ln^4 x}$$

$$x > 0$$

Ex 6) Find the solution to the differential equation assuming k is a constant

$$\frac{dy}{dt} = ky$$

$k = -0.5$, $y(0) = 200$

t y

$\frac{1}{y}\,dy = k\,dt$

$\int \frac{1}{y} \cdot dy = \int k \cdot dt$

$\ln y = k \cdot t + C$

$\ln 200 = -.5(0) + C$

$\ln 200 = C$

$\ln y = k t + C$

$e^{kt+c} = y$

$e^{kt} \cdot e^{c} = y$ $\boxed{t=0}$

$e^{k \cdot 0} \cdot e^{c} = y$

$e^{c} = y$

$y_0 = y$

$\ln y = -.5t + \ln 200$

$e^{-.5t + \ln 200} = y$

$e^{-.5t} \cdot e^{\ln 200} = y$

$\boxed{200e^{-.5t} = y}$

Law of exponential Change

$y = y_0 e^{kt}$

$y = Ae^{kt}$

$y = Pe^{kt}$

$y = e^{c} e^{kt}$

Ex 7) Find the solution to the differential equation assuming k is a constant

$$\frac{dy}{dt} = ky \qquad y(0) = 60,\ y(10) = 30$$

$\frac{1}{y} \cdot dy = k \cdot dt$

$\int \frac{1}{y} dy = \int k \cdot dt$

$\ln y = kt + C$

$e^{kt+C} = y$

$e^{kt} \cdot e^{C} = y$ — Finding e^{C}

$t=0,\ y=60$

$e^{(k \cdot 0)} \cdot e^{C} = 60$

$e^{C} = 60$

$60e^{kt} = y$

$60e^{(k \cdot 10)} = 30$

$e^{10k} = \frac{1}{2}$

$\ln\frac{1}{2} = 10k$

$\frac{\ln\frac{1}{2}}{10} = k$

$y = 60e^{\left(\frac{\ln\frac{1}{2}}{10} \cdot t\right)}$

or

$\frac{\ln\frac{1}{2}}{10} = \frac{\ln 1 - \ln 2}{10}$

$= \frac{-\ln 2}{10}$

$y = 60e^{\left(\frac{-\ln 2}{10} t\right)}$