Key

7.1 INTEGRAL AS NET CHANGE

Distance versus Displacement

We have already seen how the position of an object can be found by finding the integral of the velocity function. The change in position is a *displacement*. To see the difference between distance and displacement, consider the following saying:

"Two steps forward and one step back"

What is the total distance traveled? 3 What is the total displacement? 1

To find displacement, we only need to find $\int_a^b v(t)\,dt$.

In order to find your new location, we say that your *new position* = *initial position* + *displacement.*

To find total distance we use $\int_a^b |v(t)|\,dt$ or find when the object is moving in the negative direction, break the integral into pieces and subtract the value of the integral for the area under the curve.

Example: Suppose the velocity of a particle moving along the $x-$axis is given by $v(t) = 6t^2 - 18t + 12$ when $0 \le t \le 2$.

a) When is the particle moving to the right? ($v+$) When is it moving left? ($v-$) When is it stopped? ($v=0$)

$0 = 6t^2 - 18t + 12$
$0 = 6(t^2 - 3t + 2)$
$0 = 6(t-2)(t-1)$
$t=2 \quad t=1$

v: + on (0, 1), − on (1, 2); marks at 0, 1, 2

right: (0,1)
left: (1,2)
Stopped: t=1, t=2

b) Find the particle's displacement for the time interval.

$\int_0^2 (6t^2 - 18t + 12)\,dt = 2t^3 - 9t^2 + 12t\Big|_0^2 = (16 - 36 + 24) - 0 = 4$

c) Find the particle's total distance traveled by setting up ONE integral and using your calculator.

$\int_0^2 |(6t^2 - 18t + 12)|\,dt = \boxed{6}$

d) Find the particle's total distance traveled without using absolute value.

$\int_0^1 (6t^2 - 18t + 12)\,dt - \int_1^2 (6t^2 - 18t + 12)\,dt$

$2t^3 - 9t^2 + 12t\Big|_0^1 - 2t^3 - 9t^2 + 12t\Big|_1^2$

$(2 - 9 + 12) - 0 - (16 - 36 + 24 - (2 - 9 + 12))$

$5 - (4 - 5)$

$5 + 1 = \boxed{6}$

Example: a) Integrating velocity gives displacement

b) Integrating the absolute value of velocity gives distance.

Consumption over Time

Velocity is not the only rate in which you can integrate to get a total. In fact if you were given a function that gave the number of tickets per hour that the police wrote each day, and you wanted to find the total number of tickets in a 24-hour period, you could integrate.

Example: [2005 AP Calculus AB #2 … Calculator Allowed] The tide removes sand from Sandy Point Beach at a rate modeled by the function R given by

$$R(t)=2+5\sin\left(\frac{4\pi t}{25}\right).$$

A pumping station adds sand to the beach at a rate modeled by the function S, given by

$$S(t)=\frac{15t}{1+3t}.$$

Both $R(t)$ and $S(t)$ have units of $\boxed{\text{cubic yards per hour}}$ and t is measured in hours for $0 \le t \le 6$. At time $t = 0$, the beach contains 2500 cubic yards of sand.

a) How much sand will the tide remove from the beach during this 6-hour period? Indicate units of measure.

$u = \frac{4\pi t}{25}$

$du = \frac{4\pi}{25}dt$

$\frac{25}{4\pi} \cdot du = dt$

$\int_0^6 \left(2 + 5\sin\left(\frac{4\pi t}{25}\right)\right)dt$

$\int_0^6 2 + \frac{5 \cdot 25}{4\pi}\int \sin u\, du$

$2t + -\frac{125}{4\pi} \cdot \cos\left(\frac{4\pi t}{25}\right)\Big|_0^6$

$2(6) + 9.86874 - \left(0 - \frac{125}{4\pi}\right)$

$= \boxed{31.8159 \text{ yds}^3 \text{ of Sand}}$

b) Write an expression for $Y(t)$, the total number of cubic yards of sand on the beach at time t.

$y(t) = 2500 + \int_0^t (S(x) - R(x))dx$

*Fundamental Thm of Calculus

c) Find the rate at which the total amount of sand on the beach is changing at time $t = 4$.

$y'(t) = S(t) - R(t)$

$y'(4) = S(4) - R(4) = \boxed{-1.909 \text{ yd}^3/\text{hr}}$

$S(4) = \frac{15 \cdot 4}{1+3 \cdot 4} = 4.61538$

$R(4) = 2 + 5\sin\left(\frac{4\pi \cdot 4}{25}\right) = 6.524135$

d) For $0 \le t \le 6$, at what time t is the amount of sand on the beach a minimum? What is the minimum value? Justify your answers.

$y'(t) = 0$ when $S(t) - R(t) = 0$, only value in $[0,6]$ to satisfy $S(t) = R(t)$ is $a = 5.117865$

** Graph $S(t) + R(t)$ + find intersection pt.

t	$y(t)$
0	2500
a	2492.3694
6	2493.2766

Amt of Sand is a minimum when $\boxed{t = 5.118 \text{ hrs}}$ + $\boxed{\text{minimum value is } 2492.369 \text{ yd}^3}$

$y(5.118) = 2500 + \int_0^{5.118} (S(x) - R(x))dx$

$= 2500 + -7.6305 = 2492.3695$

Example: [2003 AP Calculus AB #2 … Calculator Allowed] A particle moves along the x – axis so that its velocity at time t is given by

$$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right).$$

At time $t = 0$, the particle is at position $x = 1$.

Since $a(2)>0$ and $v(2)<0$, speed is decreasing.

a) Find the acceleration of the particle at time $t = 2$. Is the speed of the particle increasing at $t = 2$? Why or why not?

$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right)$

$= (-t-1)\sin\left(\frac{t^2}{2}\right)$

$vu' + uv'$

$a(t) = \sin\left(\frac{t^2}{2}\right)(-1) + (-t-1)\cos\left(\frac{t^2}{2}\right)\cdot t = \boxed{-\sin\left(\frac{t^2}{2}\right) - t(t+1)\cos\left(\frac{t^2}{2}\right)}$

$a(2) = \boxed{1.5875}$

$v(2) = -3\sin(2) < 0$

b) Find all times t in the open interval $0 < t < 3$ when the particle changes direction. Justify your answer.

$v(t) = -(t+1)\sin\left(\frac{t^2}{2}\right) = 0$

$t+1=0$, $t=-1$

$\sin\frac{t^2}{2}=0$

$\frac{t^2}{2}=0$, $t=0$ | $\frac{t^2}{2}=\pi$, $t^2=2\pi$, $t=\sqrt{2\pi}$ | $\cancel{\frac{t^2}{2}=2\pi}$, $\cancel{t^2=4\pi}$

$-$ $+$

-1 0 $\sqrt{2\pi}$ 3

2.5

$\boxed{t=\sqrt{2\pi}}$

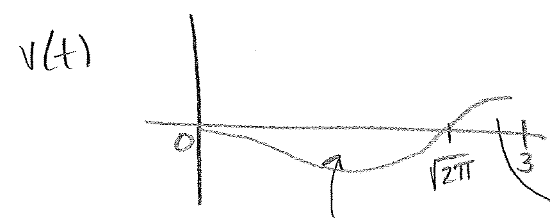
$v(t)$ switches from $-$ to $+$.

c) Find the total distance traveled by the particle from time $t = 0$ until time $t = 3$.

$\int_0^3 \left|-(t+1)\sin\left(\frac{t^2}{2}\right)\right| dt = \boxed{4.3378}$

d) During the time interval $0 \le t \le 3$, what is the greatest distance between the particle and the origin? Show the work that leads to your answer.

$|-2.265| = \boxed{2.265}$



$t=\sqrt{2\pi}$ $t=0$

$x=-2.265$ 0 $x=1$

$t=3$

-1.196 0

Area

$\int_0^{\sqrt{2\pi}} v(t) = \boxed{-3.265}$

Area

$\int_{\sqrt{2\pi}}^3 v(t) = \boxed{1.06833}$

Example: [2004 AP Calculus AB (Form B) #2 … Calculator Allowed] For $0 \le t \le 31$, the rate of change of the number of mosquitoes on Tropical Island at time t days is modeled by $R(t) = 5\sqrt{t}\cos\left(\frac{t}{5}\right)$ mosquitoes per day. There are 1000 mosquitoes on Tropical Island at time $t = 0$.

a) Show that the number of mosquitoes is increasing at time $t = 6$.

$\boxed{R(6) = 4.438 > 0}$ the number of mosquitos is increasing at $t=6$.

b) At time $t = 6$, is the number of mosquitoes increasing at an increasing rate, or is the number of mosquitoes increasing at a decreasing rate? Give a reason for your answer.

$\boxed{R'(6) = -1.913}$ $R'(6) < 0$, # of mosquitos is increasing at a decreasing rate at $t=6$.

$vu' + uv'$

$R(t) = \underbrace{5\sqrt{t}}_{u} \cdot \underbrace{\cos\left(\frac{t}{5}\right)}_{v}$

$R'(t) = \cos\left(\frac{t}{5}\right)\cdot 5\cdot\frac{1}{2}\cdot t^{-1/2} + 5\sqrt{t}\cdot -\sin\left(\frac{t}{5}\right)\cdot\frac{1}{5}$ ⟸ Plug in 6

c) According to the model, how many mosquitoes will be on the island at time $t = 31$? Round your answer to the nearest whole number.

$1000 + \int_0^{31} R(t)\,dt = 964.335 \approx \boxed{964 \text{ mosiquitos}}$

use calc

☺

d) To the nearest whole number, what is the maximum number of mosquitoes for $0 \le t \le 31$? Show the analysis that leads to your conclusion.

$R(t) = 0$ when $t = 0$, $t = 2.5\pi$ or $t = 7.5\pi$

$R(t) > 0$ on $0 < t < 2.5\pi$

$R(t) < 0$ on $2.5\pi < t < 7.5\pi$

$R(t) > 0$ on $7.5\pi < t < 31$

graph and find zeros!

or

R: + on $(0, 2.5\pi)$, − on $(2.5\pi, 7.5\pi)$, + on $(7.5\pi, 31)$

$5\sqrt{t}\cos\frac{t}{5} = 0$

$\sqrt{t} = 0$, $\boxed{t = 0}$ | $\frac{t}{5} = \frac{\pi}{2}$, $t = \frac{5\pi}{2}$ | $\frac{t}{5} = \frac{3\pi}{2}$, $t = 7.5\pi$

The absolute max # of mosquitos is at $t = 2.5\pi$ or $t = 31$.

$1000 + \int_0^{2.5\pi} R(t) = 1039.351$

so the max is $\boxed{1039 \text{ mosquitos}}$

since at $31 = t$ there are 964 mosiquitos