Worksheet 2. Solutions

Example 1 (numerical)

1. At $t = 0$, the particle is moving to the left because the velocity is negative.

2. Yes, there is a time when the particle is at rest during the time interval $0 < t < 12$ minutes. Since the velocity function is differentiable, it also is continuous. Hence, by the Intermediate Value Theorem, since velocity goes from negative to positive, it must go through zero and $v(t) = 0$ means the particle is at rest.

3. Since $t = 10$ is not one of the times given in the table, we should approximate the derivative by using a difference quotient with the best (closest) data available. Because 10 lies between 8 and 12, the best approximation is given by:

$$v'(10) \approx \frac{v(12) - v(8)}{12 - 8} = \frac{5 - 7}{12 - 8} = -\frac{1}{2}\ \frac{\text{m}/\text{min}}{\text{min}} = \frac{\text{m}}{\text{min}^2}.$$

   Here, $v'(10)$ is the acceleration of the particle at $t = 10$ minutes.

4. Yes, such a point is guaranteed by the Mean Value Theorem or Rolle's Theorem. Since velocity is differentiable (and therefore also continuous) over the interval $6 < t < 12$ and:

$$\frac{v(12) - v(6)}{12 - 6} = 0,$$

   then there must exist a point c in the interval such that $v'(c) = a(c) = 0$.

   Note: If we add the hypothesis that v' is continuous, then we may use the Intermediate Value Theorem to establish the result. In this case, since the values in the table indicate that velocity increases and then decreases on the interval $0 < t < 12$, then $v'(t) = a(t)$ must go from positive to negative and by the Intermediate Value Theorem must therefore pass through zero somewhere in that interval. It is the Mean Value Theorem, applied to the differentiable function v, that guarantees v' takes on at least one positive value in the interval $0 < t < 8$ (note that $\frac{7 - (-3)}{8 - 0} = \frac{5}{4}$ is one such value), and at least one negative value in the interval $8 < t < 12$ (note that $\frac{5 - 7}{12 - 8} = -\frac{1}{2}$ is one such value).

Example 2 (graphical)

1. At $t = 4$ seconds, the particle is moving to the right because the velocity is positive.

2. The particle is moving to the left over the interval $5 < t \leq 9$ seconds because the velocity is negative.

3. The acceleration of the particle is negative because the velocity is decreasing, OR the acceleration is the slope of the velocity graph and the slope of the velocity graph at $t =$ 4 is negative.

4. Average acceleration over the time interval can be found by dividing the change in velocity by the change in time:

$$\frac{v(4)-v(2)}{4-2} = \frac{6-9}{4-2} = -\frac{3}{2}\ \frac{\text{ft}/\text{sec}}{\text{sec}} = \frac{\text{ft}}{\text{sec}^2}$$

5. No such time is guaranteed. The Mean Value Theorem does not apply since the function is not differentiable at $t = 3$ due to the sharp turn in the graph. If students have not yet learned the MVT, you can slide a tangent line (toothpick or stick) along the graph to show that no such point exists where the slope of the tangent line would be equal to the slope of a secant line between $t = 2$ and $t = 4$.

6. The particle is farthest to the right at $t = 5$ seconds. Since the velocity is positive during the time interval $0 \leq t < 5$ seconds and negative during the time interval $5 < t \leq 9$ seconds, the particle moves to the right before time $t = 5$ seconds and moves to the left after that time. Therefore, it is farthest to the right at $t = 5$ seconds.

Example 3 (analytic)

1. The particle is moving to the right because $x'(0) = v(0) = 9$ which is positive.

2. At $t = 1$, the velocity of the particle is decreasing because $x''(1) = v'(1) = a(1) = -6$, and if the acceleration is negative then the velocity is decreasing.

3. The particle is moving to the left for all values of t where $v(t) < 0$. We have:

$$v(t) = x'(t) = 3t^2 - 12t + 9 < 0 \text{ for } 1 < t < 3.$$

4.

t	$x(t)$
0	11
1	15
3	11
5	31

The particle moves right 4 spaces, left 4 spaces and then right 20 spaces. Therefore, the particle has traveled a total of 28 units. The common error that students make is to calculate $x(5) - x(0) = 20$, which gives the displacement or net change in position, rather than the total distance traveled. Teachers should reinforce the difference between these two concepts every chance they get.

Dixie Ross, Pflugerville High School, Pflugerville, Texas