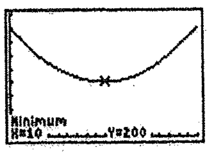
Section 4.4 Exercises

1. Represent the numbers by x and $20-x$, where $0 \le x \le 20$.

(a) The sum of the squares is given by $f(x) = x^2 + (20-x)^2 = 2x^2 - 40x + 400$. Then $f'(x) = 4x - 40$. The critical point and endpoints occur at $x = 0, x = 10$, and $x = 20$. Then $f(0) = 400, f(10) = 200$, and $f(20) = 400$. The sum of the squares is as large as possible for the numbers 0 and 20, and is as small as possible for the numbers 10 and 10.

Graphical support:

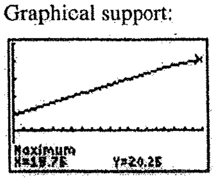


[0, 20] by [0, 450]

(b) The sum of one number plus the square root of the other is given by $g(x) = x + \sqrt{20-x}$. Then

$g'(x) = 1 - \frac{1}{2\sqrt{20-x}}$. The critical point occurs when $2\sqrt{20-x} = 1$, so $20 - x = \frac{1}{4}$ and $x = \frac{79}{4}$. Testing the endpoints and critical point, we find $g(0) = \sqrt{20} \approx 4.47$, $g\left(\frac{79}{4}\right) = \frac{81}{4} = 20.25$, and $g(20) = 20$. The sum is as large as possible when the numbers are $\frac{79}{4}$ and $\frac{1}{4}$ $\left(\text{summing } \frac{79}{4} + \sqrt{\frac{1}{4}}\right)$, and is as small as possible when the numbers are 0 and 20 (summing $0 + \sqrt{20}$).

Graphical support:



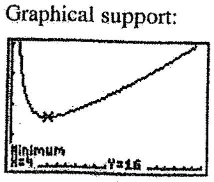
[0, 20] by [−10, 25]

3. Let x represent the length of the rectangle in inches ($x > 0$). Then the width is $\frac{16}{x}$ and the perimeter is

$P(x) = 2\left(x + \frac{16}{x}\right) = 2x + \frac{32}{x}$.

Since $P'(x) = 2 - 32x^{-2} = \frac{2(x^2-16)}{x^2}$ this critical point occurs at $x = 4$. Since $P'(x) < 0$ for $0 < x < 4$ and $P'(x) > 0$ for $x > 4$, this critical point corresponds to the minimum perimeter. The smallest possible perimeter is $P(4) = 16$ in., and the rectangle's dimensions are 4 in. by 4 in.

Graphical support:



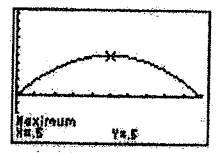
[0, 20] by [0, 40]

5. (a) The equation of line AB is $y = -x + 1$, so the y-coordinate of P is $-x + 1$.

(b) $A(x) = 2x(1-x)$

(c) Since $A'(x) = \frac{d}{dx}(2x - 2x^2) = 2 - 4x$, the critical point occurs at $x = \frac{1}{2}$. Since $A'(x) > 0$ for $0 < x < \frac{1}{2}$ and $A'(x) < 0$ for $- < x < 1$, this critical point corresponds to the maximum area. The largest possible area is $A\left(\frac{1}{2}\right) = \frac{1}{2}$ square unit, and the dimensions of the rectangle are $\frac{1}{2}$ unit by 1 unit.

Graphical support:



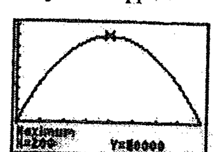
[0, 1] by [−0.5, 1]

9. Let x be the length in meters of each side that adjoins the river. Then the side parallel to the river measures $800 - 2x$ meters and the area is

$A(x) = x(800 - 2x) = 800x - 2x^2$ for $0 < x < 400$.

Therefore, $A'(x) = 800 - 4x$ and the critical point occurs at $x = 200$. Since $A'(x) > 0$ for $0 < x < 200$ and $A'(x) < 0$ for $200 < x < 400$, the critical point corresponds to the maximum area. The largest possible area is $A(200) = 80{,}000$ m^2 and the dimensions are 200 m (perpendicular to the river) by 400 m (parallel to the river).

Graphical support:



[0, 400] by [−25,000, 90,000]

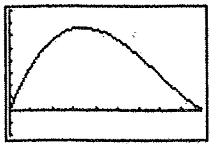
13. Let x be the height in inches of the printed area. Then the width of the printed area is $\frac{50}{x}$ in. and the overall dimensions are $x + 8$ in. by $\frac{50}{x} + 4$ in. The amount of paper used is $A(x) = (x+8)\left(\frac{50}{x} + 4\right) = 4x + 82 + \frac{400}{x}$ in^2. Then

$A'(x) = 4 - 400x^{-2} = \frac{4(x^2 - 100)}{x^2}$ and the critical point (for $x > 0$) occurs at $x = 10$. Since $A'(x) < 0$ for $0 < x < 10$ and $A'(x) > 0$ for $x > 10$, the critical point corresponds to the minimum amount of paper. Using $x + 8$ and $\frac{50}{x} + 4$ for $x = 10$, the overall dimensions are 18 in. high by 9 in. wide.

19. (a) The "sides" of the suitcase will measure $24 - 2x$ in. by $18 - 2x$ in. and will be $2x$ in. apart, so the volume formula is

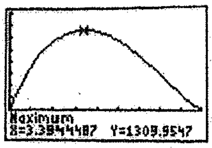
$V(x) = 2x(24-2x)(18-2x) = 8x^3 - 168x^2 + 864x$.

(b) We require $x > 0, 2x < 18$, and $2x < 24$. Combining these requirements, the domain is the interval (0, 9).



[0, 9] by [−400, 1600]

(c)



[0, 9] by [−400, 1600]

The maximum volume is approximately 1309.95 when $x \approx 3.39$ in.

(d) $V'(x) = 24x^2 - 336x + 864 = 24(x^2 - 14x + 36)$

The critical point is at

$$x = \frac{4 \pm \sqrt{(-14)^2 - 4(1)(36)}}{2(1)} = \frac{14 \pm \sqrt{52}}{2} = 7 \pm \sqrt{13},$$ that

is, $x \approx 3.39$ or $x \approx 10.61$. We discard the larger value because it is not in the domain. Since $V''(x) = 24(2x - 14)$, which is negative when $x \approx 3.39$, the critical point corresponds to the maximum volume. The maximum value occurs at $x = 7 - \sqrt{13} \approx 3.39$, which confirms the results in (c).

(e) $8x^3 - 168x^2 + 864x = 1120$

$8(x^3 - 21x^2 + 108x - 140) = 0$

$8(x-2)(x-5)(x-14) = 0$

Since 14 is not in the domain, the possible values of x are $x = 2$ in. or $x = 5$ in.

(f) The dimensions of the resulting box are $2x$ in., $(24 - 2x)$ in., and $(18 - 2x)$ in. Each of these measurements must be positive, so that gives the domain of (0, 9).

21. If the upper right vertex of the rectangle is located at $(x, 4\cos 0.5x)$ for $0 < x < \pi$, then the rectangle has width $2x$ and height $4\cos 0.5x$, so the area is $A(x) = 8x\cos 0.5x$.

Then $A'(x) = 8x(-0.5\sin 0.5x) + 8(\cos 0.5x)(1)$

$= -4x\sin 0.5x + 8\cos 0.5x$.

Solving $A'(x)$ graphically for $0 < x < \pi$, we find that $x \approx 1.72$. Evaluating $2x$ and $4\cos 0.5x$ for $x \approx 1.72$, the dimensions of the rectangle are approximately 3.44 (width) by 2.61 (height), and the maximum area is approximately 8.98.

25. Set $c'(x) = \frac{c(x)}{x}$: $3x^2 - 20x + 30 = x^2 - 10x + 30$. The only positive solution is $x = 5$, so average cost is minimized at a production level of 5000 units. Note that

$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$ for all positive x, so the Second Derivative Test Confirms the minimum.

33. (a) We require $f(x)$ to have a critical point at $x = 2$. Since $f'(x) = 2x - ax^{-2}$, we have $f'(2) = 4 - \frac{a}{4}$ and so our requirement is that $4 - \frac{a}{4} = 0$. Therefore, $a = 16$. To verify that the critical point corresponds to a local minimum, note that we now have $f'(x) = 2x - 16x^{-2}$ and so $f''(x) = 2 + 32x^{-3}$, so $f''(2) = 6$, which is positive as expected. So, use $a = -16$.

(b) We require $f''(1) = 0$. Since $f'' = 2 + 2ax^{-3}$, we have $f''(1) = 2 + 2a$, so our requirement is that $2 + 2a = 0$. Therefore, $a = -1$. To verify that $x = 1$ is in fact an inflection point, note that we now have $f''(x) = 2 - 2x^{-3}$, which is negative for $0 < x < 1$ and positive for $x > 1$. Therefore, the graph of f is concave down in the interval (0, 1) and concave up in the interval $(1, \infty)$. So, use $a = -1$.

41. The square of the distance is

$$D(x) = \left(x - \frac{3}{2}\right)^2 + (\sqrt{x} + 0)^2 = x^2 - 2x + \frac{9}{4},$$

so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

47. The trapezoid has height $(\cos\theta)$ft and the trapezoid bases measure 1 ft and $(1 + 2\sin\theta)$ft, so the volume is given by

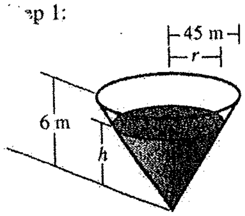
$$V(\theta) = \frac{1}{2}(\cos\theta)(1 + 1 + 2\sin\theta)(20)$$
$$= 20(\cos\theta)(1 + \sin\theta).$$

Find the critical points for $0 \le \theta < \frac{\pi}{2}$:

$$V'(\theta) = 20(\cos\theta)(\cos\theta) + 20(1 + \sin\theta)(-\sin\theta) = 0$$
$$20\cos^2\theta - 20\sin\theta - 20\sin^2\theta = 0$$
$$20(1 - \sin^2\theta) - 20\sin\theta - 20\sin^2\theta = 0$$
$$-20(2\sin^2\theta + \sin\theta - 1) = 0$$
$$-20(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2} \text{ or } \sin\theta = -1$$
$$\theta = \frac{\pi}{6}$$

The critical point is at $\left(\frac{\pi}{6}, 15\sqrt{3}\right)$. Since $V'(\theta) > 0$ for $0 \le \theta < \frac{\pi}{6}$ and $V'(\theta) < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$, the critical point corresponds to the maximum possible trough volume. The volume is maximized when $\theta = \frac{\pi}{6}$.

17. Step 1:



$r =$ radius of top surface of water
$h =$ depth of water in reservoir
$V =$ volume of water in reservoir

Step 2:

At the instant in question, $\frac{dV}{dt} = -50 \text{ m}^3/\text{min}$ and $h = 5$ m.

Step 3:

We want to find $-\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Step 4:

Note that $\frac{h}{r} = \frac{6}{45}$ by similar cones, so $r = 7.5h$.

Then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(7.5h)^2 h = 18.75\pi h^3$

Steps 5 and 6:

(a) Since $V = 18.75\pi h^3, \frac{dV}{dt} = 56.25\pi h^2 \frac{dh}{dt}$.

Thus $-50 = 56.25\pi(5^2)\frac{dh}{dt}$, and

so $\frac{dh}{dt} = -\frac{8}{225\pi}$ m/min $= -\frac{32}{9\pi}$ cm/min.

The water level is falling by $\frac{32}{9\pi} \approx 1.13$ cm/min.

(Since $\frac{dh}{dt} < 0$, the rate at which the water level is *falling* is positive.)

(b) Since $r = 7.5h, \frac{dr}{dt} = 7.5\frac{dh}{dt} = -\frac{80}{3\pi}$ cm/min. The rate of change of the radius of the water's surface is $-\frac{80}{3\pi} \approx -8.49$ cm/min.

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19. Step 1:

$x =$ distance from wall to base of ladder
$y =$ height of top of ladder
$A =$ area of triangle formed by the ladder, wall, and ground
θ = angle between the ladder and the ground

Step 2:

At the instant in question, $x = 12$ ft and $\frac{dx}{dt} = 5$ ft/sec.

Step 3:

We want to find $-\frac{dy}{dy}, \frac{dA}{dt}$, and $\frac{d\theta}{dt}$.

Steps 4, 5, and 6:

(a) $x^2 + y^2 = 169$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

To evaluate, note that, at the instant in question,

$$y = \sqrt{169 - x^2} = \sqrt{169 - 12^2} = 5.$$

Then $2(12)(5) + 2(5)\frac{dy}{dt} = 0$

$$\frac{dy}{dt} = -12 \text{ ft/sec} \left(\text{or } -\frac{dy}{dt} = 12 \text{ ft/sec}\right)$$

The top of the ladder is sliding down the wall at the rate of 12 ft/sec. (Note that the *downward* rate of motion is positive.)

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

Using the results from step 2 and from part (a), we have $\frac{dA}{dt} = \frac{1}{2}[(12)(-12) + (5)(5)] = -\frac{119}{2}$ ft^2/sec. The area of the triangle is changing at the rate of -59.5 ft^2/sec.

(c) $\tan\theta = \frac{y}{x}$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

Since $\tan\theta = \frac{5}{12}$, we have

$$\left(\text{for } 0 \le \theta < \frac{\pi}{2}\right)\cos\theta = \frac{12}{13} \text{ and so } \sec^2\theta \frac{1}{\left(\frac{12}{13}\right)^2} = \frac{169}{144}.$$

Combining this result with the results from step 2 and from part (a), we have $\frac{169}{144}\frac{d\theta}{dt} = \frac{(12)(-12) - (5)(5)}{12^2}$, so $\frac{d\theta}{dt} = -1$ radian/sec. The angle is changing at the rate of -1 radian/sec.

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23. $\frac{dy}{dt} = \frac{dy}{dt}\frac{dx}{dt} = -10(1 + x^2)^{-2}(2x)\frac{dx}{dt} = -\frac{20x}{(1 + x^2)}\frac{dx}{dt}$

Since $\frac{dx}{dt} = 3$cm/sec, we have

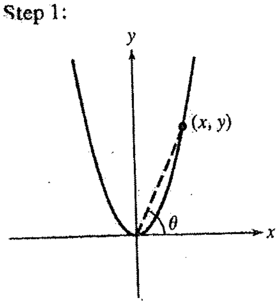
$$\frac{dy}{dt} = -\frac{60x}{(1 + x^2)^2} \text{ cm/sec.}$$

(a) $\frac{dy}{dt} = -\frac{60(-2)}{[1 + (-2)^2]^2} = \frac{120}{5^2} = \frac{24}{5}$ cm/sec

(b) $\frac{dy}{dt} = -\frac{60(0)}{(1 + 0^2)^2} = 0$ cm/sec

(c) $\frac{dy}{dt} = -\frac{60(20)}{(1 + 20^2)^2} \approx -0.00746$ cm/sec

25. Step 1:



x = x-coordinate of particle's location
y = y-coordinate of particle's location
θ = angle of inclination of line joining the particle to the origin.

Step 2:
At the instant in question,
$\frac{dx}{dt} = 10$ m/sec and $x = 3$ m.

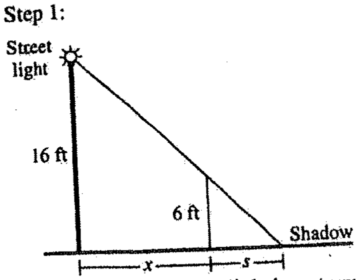
Step 3:
We want to find $\frac{d\theta}{dt}$.

Step 4:
Since $y = x^2$, we have $\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$ and so, for $x > 0$,
$\theta = \tan^{-1} x$.

Step 5:
$$\frac{d\theta}{dt} = \frac{1}{1+x^2}\frac{dx}{dt}$$

Step 6:
$$\frac{d\theta}{dt} = \frac{1}{1+3^2}(10) = 1 \text{ radian/sec}$$
The angle of inclination is increasing at the rate of 1 radian/sec.

29. Step 1:



x = distance from streetlight base to man
s = length of shadow

Step 2:
At the instant in question, $\frac{dx}{dt} = -5$ ft/sec and $x = 10$ ft.

Step 3:
We want to find $\frac{ds}{dt}$.

Step 4:
By similar triangles, $\frac{s}{6} = \frac{s+x}{16}$. This is equivalent to $16s = 6s + 6x$, or $s = \frac{3}{5}x$.

Step 5:
$$\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt}$$

27. Step 1:
r = radius of balls plus ice
S = surface area of ball plus ice
V = volume of ball plus ice

Step 2:
At the instant in question,
$\frac{dV}{dt} = -8$ mL/min $= -8\,\text{cm}^3$/min and $r = \frac{1}{2}(20) = 10$ cm.

Step 3:
We want to find $-\frac{dS}{dt}$.

Step 4:
We have $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. These equations can be combined by noting that $r = \left(\frac{3V}{4\pi}\right)^{1/3}$, so $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$

Step 5:
$$\frac{dS}{dt} = 4\pi\left(\frac{2}{3}\right)\left(\frac{3V}{4\pi}\right)^{-1/3}\left(\frac{3}{4\pi}\right)\frac{dV}{dt} = 2\left(\frac{3V}{4\pi}\right)^{-1/3}\frac{dV}{dt}$$

Step 6:
Note that $V = \frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$.
$$\frac{dS}{dt} = 2\left(\frac{3}{4\pi}\cdot\frac{4000\pi}{3}\right)^{-1/3}(-8) = \frac{-16}{\sqrt[3]{1000}} = -1.6\,\text{cm}^2/\text{min}.$$
Since $\frac{dS}{dt} < 0$, the rate of *decrease* is positive. The surface area is decreasing at the rate of $1.6\,\text{cm}^2$/min.

31. Step 1:
x = position of car ($x = 0$ when car is right in front of you)
θ = camera angle. (We assume θ is negative until the car passess in front of you, and then positive.)

Step 2:
At the first instant in question, $x = 0$ ft and $\frac{dx}{dt} = 264$ ft/sec.

A half second later, $x\frac{1}{2}(264) = 132$ ft and $\frac{dx}{dt} = 264$ ft/sec.

Step 3:
We want to find $\frac{d\theta}{dt}$ at each of the two instants.

Step 4:
$$\theta = \tan^{-1}\left(\frac{x}{132}\right)$$

Step 5:
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{132}\right)^2}\cdot\frac{1}{132}\frac{dx}{dt}$$

Step 6:
When $x = 0$: $\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{0}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 2$ radians/sec

When $x = 132$: $\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{132}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 1$ radians/sec

33. Step 1:

s = shadow length

θ = sun's angle of elevation

Step 2:

At the instant in question,

$s = 60$ ft and $\frac{d\theta}{dt} = 0.27° / \text{min} = 0.0015\pi$ radian/min.

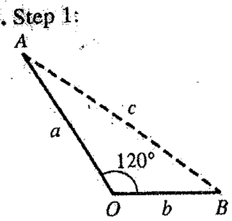
Step 3:

We want to find $-\frac{ds}{dt}$.

Step 4:

$\tan\theta = \frac{80}{s}$ or $s = 80\cot\theta$

35. Step 1:



a = distance from O to A

b = distance from O to B

c = distance from A to B

Step 2:

At the instant in question, $a = 5$ nautical miles, $b = 3$ nautical miles, $\frac{da}{dt} = 14$ knots, and $\frac{db}{dt} = 21$ knots.

Step 3:

We want to find $\frac{dc}{dt}$,

Step 4:

Law of Cosines : $c^2 = a^2 + b^2 - 2ab\cos 120°$

$$c^2 = a^2 + b^2 + ab$$

Step 5:

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}$$

Step 6:

Note that, at the instant in question,

$$c = \sqrt{a^2 + b^2 + ab} = \sqrt{(5)^2 + (3)^2 + (5)(3)} = \sqrt{49} = 7$$

$$2(7)\frac{dc}{dt} = 2(5)(14) + 2(3)(21) + (5)(21) + (3)(14)$$

$$14\frac{dc}{dt} = 413$$

$$\frac{dc}{dt} = 29.5 \text{ knots}$$

The ships are moving apart at a rate of 29.5 knots.