Section 4.6 Exercises

1. Since $\frac{dA}{dt}=\frac{dA}{dr}\frac{dr}{dt}$, we have $\frac{dA}{dt}=2\pi r\frac{dr}{dt}$.

3. (a) Since $\frac{dV}{dt}=\frac{dV}{dh}\frac{dh}{dt}$, we have $\frac{dV}{dt}=\pi r^2\frac{dh}{dt}$.

(b) Since $\frac{dV}{dt}=\frac{dV}{dr}\frac{dr}{dt}$, we have $\frac{dV}{dt}=2\pi rh\frac{dr}{dt}$.

(c) $\frac{dV}{dt}=\frac{d}{dt}\pi r^2h=\pi\frac{d}{dt}(r^2h)$

$\frac{dV}{dt}=\pi\left(r^2\frac{dh}{dt}+h(2r)\frac{dr}{dt}\right)$

$\frac{dV}{dt}=\pi r^2\frac{dh}{dt}+2\pi rh\frac{dr}{dt}$

9. Step 1:
l = length of rectangle
w = width of rectangle
A = area of rectangle
P = perimeter of rectangle
D = length of a diagonal of the rectangle

Step 2:
At the instant in question,
$\frac{dl}{dt}=-2$ cm/sec, $\frac{dw}{dt}=2$ cm/sec, $l=12$ cm, and $w=5$ cm.

Step 3:
We want to find $\frac{dA}{dt},\frac{dP}{dt}$, and $\frac{dD}{dt}$.

Steps 4, 5, and 6:

(a) $A=lw$

$\frac{dA}{dt}=l\frac{dw}{dt}+w\frac{dl}{dt}$

$\frac{dA}{dt}=(12)(2)+(5)(-2)=14\text{ cm}^2/\text{sec}$

The rate of change of the area is $14\text{ cm}^2/\text{sec}$.

(b) $P=2l+2w$

$\frac{dP}{dt}=2\frac{dl}{dt}+2\frac{dw}{dt}$

$\frac{dP}{dt}=2(-2)+2(2)=0$ cm/sec

The rate of change of the perimeter is 0 cm/sec.

(c) $D=\sqrt{l^2+w^2}$

$$\frac{dD}{dt}=\frac{1}{2\sqrt{l^2+w^2}}\left(2l\frac{dl}{dt}+2w\frac{dw}{dt}\right)=\frac{l\frac{dl}{dt}+w\frac{dw}{dt}}{\sqrt{l^2+w^2}}$$

$$\frac{dD}{dt}=\frac{(12)(-2)+(5)(2)}{\sqrt{12^2+5^2}}=-\frac{14}{13}\text{ cm/sec}$$

The rate of change of the length of the diameter is $-\frac{14}{13}$ cm/sec.

(d) The area is increasing, because its derivative is positive. The perimeter is not changing, because its derivative is zero. The diagonal length is decreasing, because its derivative is negative.

11. Step 1:
r = radius of spherical balloon
S = surface area of spherical balloon
V = volume of spherical balloon

Step 2:
At the instant in question, $\frac{dV}{dt}=100\pi\text{ ft}^3/\text{min}$ and $r=5$ ft.

Step 3:
We want to find the values of $\frac{dr}{dt}$ and $\frac{dS}{dt}$.

Steps 4, 5, and 6:

(a) $V=\frac{4}{3}\pi r^3$

$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

$100\pi=4\pi(5)^2\frac{dr}{dt}$

$\frac{dr}{dt}=1$ ft/min

The radius is increasing at the rate of 1 ft/min.

(b) $S=4\pi r^2$

$\frac{dS}{dt}=8\pi r\frac{dr}{dt}$

$\frac{dS}{dt}=8\pi(5)(1)$

$\frac{dS}{dt}=40\pi\text{ ft}^2/\text{min}$

The surface area is increasing at the rate of $40\pi\text{ ft}^2/\text{min}$.

13. Step 1:
s = (diagonal) distance from antenna to airplane
x = horizontal distance from antenna to airplane

Step 2:
At the instant in question,
$s=10$ mi and $\frac{ds}{dt}=300$ mph.

Step 3:
We want to find $\frac{dx}{dt}$.

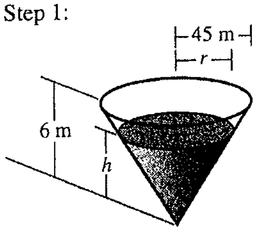
Step 4:
$x^2+49=s^2$ or $x=\sqrt{s^2-49}$

Step 5:
$$\frac{dx}{dt}=\frac{1}{2\sqrt{s^2-49}}\left(2s\frac{ds}{dt}\right)=\frac{s}{\sqrt{s^2-49}}\frac{dt}{dt}$$

Step 6:
$$\frac{dx}{dt}=\frac{10}{\sqrt{10^2-49}}(300)=\frac{3000}{\sqrt{51}}\text{ mph}\approx 420.08\text{ mph}$$

The speed of the airplane is about 420.08 mph.

17. Step 1:



r = radius of top surface of water
h = depth of water in reservoir
V = volume of water in reservoir

Step 2:

At the instant in question, $\frac{dV}{dt} = -50\ \text{m}^3/\text{min}$ and $h = 5$ m.

Step 3:

We want to find $-\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Step 4:

Note that $\frac{h}{r} = \frac{6}{45}$ by similar cones, so $r = 7.5h$.

Then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(7.5h)^2 h = 18.75\pi h^3$

Steps 5 and 6:

(a) Since $V = 18.75\pi h^3, \frac{dV}{dt} = 56.25\pi h^2 \frac{dh}{dt}$.

Thus $-50 = 56.25\pi(5^2)\frac{dh}{dt}$, and

so $\frac{dh}{dt} = -\frac{8}{225\pi}$ m/min $= -\frac{32}{9\pi}$ cm/min.

The water level is falling by $\frac{32}{9\pi} \approx 1.13$ cm/min.

(Since $\frac{dh}{dt} < 0$, the rate at which the water level is *falling* is positive.)

(b) Since $r = 7.5h, \frac{dr}{dt} = 7.5\frac{dh}{dt} = -\frac{80}{3\pi}$ cm/min. The rate of change of the radius of the water's surface is $-\frac{80}{3\pi} \approx -8.49$ cm/min.

19. Step 1:

x = distance from wall to base of ladder
y = height of top of ladder
A = area of triangle formed by the ladder, wall, and ground
θ = angle between the ladder and the ground

Step 2:

At the instant in question, $x = 12$ ft and $\frac{dx}{dt} = 5$ ft/sec.

Step 3:

We want to find $-\frac{dy}{dy}, \frac{dA}{dt}$, and $\frac{d\theta}{dt}$.

Steps 4, 5, and 6:

(a) $x^2 + y^2 = 169$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

To evaluate, note that, at the instant in question,

$$y = \sqrt{169 - x^2} = \sqrt{169 - 12^2} = 5.$$

Then $2(12)(5) + 2(5)\frac{dy}{dt} = 0$

$$\frac{dy}{dt} = -12 \text{ ft/sec} \left(\text{or } -\frac{dy}{dt} = 12 \text{ ft/sec}\right)$$

The top of the ladder is sliding down the wall at the rate of 12 ft/sec. (Note that the *downward* rate of motion is positive.)

(b) $A = \frac{1}{2}xy$

$$\frac{dA}{dt} = \frac{1}{2}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right)$$

Using the results from step 2 and from part (a), we have $\frac{dA}{dt} = \frac{1}{2}[(12)(-12) + (5)(5)] = -\frac{119}{2}$ ft^2/sec. The area of the triangle is changing at the rate of -59.5 ft^2/sec.

(c) $\tan\theta = \frac{y}{x}$

$$\sec^2\theta\frac{d\theta}{dt} = \frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2}$$

Since $\tan\theta = \frac{5}{12}$, we have

$$\left(\text{for } 0 \le \theta < \frac{\pi}{2}\right)\cos\theta = \frac{12}{13} \text{ and so } \sec^2\theta \frac{1}{\left(\frac{12}{13}\right)^2} = \frac{169}{144}.$$

Combining this result with the results from step 2 and from part (a), we have $\frac{169}{144}\frac{d\theta}{dt} = \frac{(12)(-12) - (5)(5)}{12^2}$, so $\frac{d\theta}{dt} = -1$ radian/sec. The angle is changing at the rate of -1 radian/sec.

23. $\frac{dy}{dt} = \frac{dy}{dt}\frac{dx}{dt} = -10(1 + x^2)^{-2}(2x)\frac{dx}{dt} = -\frac{20x}{(1 + x^2)}\frac{dx}{dt}$

Since $\frac{dx}{dt} = 3$cm/sec, we have

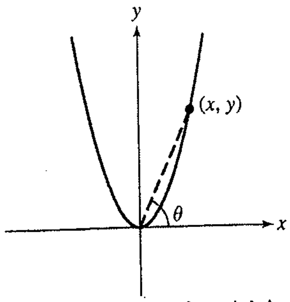
$$\frac{dy}{dt} = -\frac{60x}{(1 + x^2)^2} \text{ cm/sec.}$$

(a) $\frac{dy}{dt} = -\frac{60(-2)}{[1 + (-2)^2]^2} = \frac{120}{5^2} = \frac{24}{5}$ cm/sec

(b) $\frac{dy}{dt} = -\frac{60(0)}{(1 + 0^2)^2} = 0$ cm/sec

(c) $\frac{dy}{dt} = -\frac{60(20)}{(1 + 20^2)^2} \approx -0.00746$ cm/sec

25. Step 1:



x = x-coordinate of particle's location
y = y-coordinate of particle's location
θ = angle of inclination of line joining the particle to the origin.

Step 2:
At the instant in question,
$\frac{dx}{dt} = 10$ m/sec and $x = 3$ m.

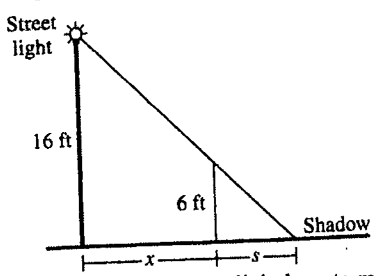
Step 3:
We want to find $\frac{d\theta}{dt}$.

Step 4:
Since $y = x^2$, we have $\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$ and so,
for $x > 0$,
$\theta = \tan^{-1} x$.

Step 5:
$$\frac{d\theta}{dt} = \frac{1}{1+x^2}\frac{dx}{dt}$$

Step 6:
$$\frac{d\theta}{dt} = \frac{1}{1+3^2}(10) = 1 \text{ radian/sec}$$
The angle of inclination is increasing at the rate of 1 radian/sec.

29. Step 1:



x = distance from streetlight base to man
s = length of shadow

Step 2:
At the instant in question, $\frac{dx}{dt} = -5$ ft/sec and $x = 10$ ft.

Step 3:
We want to find $\frac{ds}{dt}$.

Step 4:
By similar triangles, $\frac{s}{6} = \frac{s+x}{16}$. This is equivalent to
$16s = 6s + 6x$, or $s = \frac{3}{5}x$.

Step 5:
$$\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt}$$

27. Step 1:
r = radius of balls plus ice
S = surface area of ball plus ice
V = volume of ball plus ice

Step 2:
At the instant in question,
$\frac{dV}{dt} = -8$ mL/min $= -8\text{ cm}^3$/min and $r = \frac{1}{2}(20) = 10$ cm.

Step 3:
We want to find $-\frac{dS}{dt}$.

Step 4:
We have $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. These equations can be combined by noting that $r = \left(\frac{3V}{4\pi}\right)^{1/3}$, so $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$

Step 5:
$$\frac{dS}{dt} = 4\pi\left(\frac{2}{3}\right)\left(\frac{3V}{4\pi}\right)^{-1/3}\left(\frac{3}{4\pi}\right)\frac{dV}{dt} = 2\left(\frac{3V}{4\pi}\right)^{-1/3}\frac{dV}{dt}$$

Step 6:
Note that $V = \frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$.
$$\frac{dS}{dt} = 2\left(\frac{3}{4\pi}\cdot\frac{4000\pi}{3}\right)^{-1/3}(-8) = \frac{-16}{\sqrt[3]{1000}} = -1.6\text{ cm}^2/\text{min}$$
Since $\frac{dS}{dt} < 0$, the rate of *decrease* is positive. The surface area is decreasing at the rate of 1.6 cm^2/min.

31. Step 1:
x = position of car ($x = 0$ when car is right in front of you)
θ = camera angle. (We assume θ is negative until the car passess in front of you, and then positive.)

Step 2:
At the first instant in question, $x = 0$ ft and $\frac{dx}{dt} = 264$ ft/sec.
A half second later, $x \frac{1}{2}(264) = 132$ ft and $\frac{dx}{dt} = 264$ ft/sec.

Step 3:
We want to find $\frac{d\theta}{dt}$ at each of the two instants.

Step 4:
$$\theta = \tan^{-1}\left(\frac{x}{132}\right)$$

Step 5:
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{132}\right)^2}\cdot\frac{1}{132}\frac{dx}{dt}$$

Step 6:
When $x = 0$: $\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{0}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 2$ radians/sec

When $x = 132$: $\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{132}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 1$ radians/sec

33. Step 1:

s = shadow length

θ = sun's angle of elevation

Step 2:

At the instant in question,

$s = 60$ ft and $\frac{d\theta}{dt} = 0.27°/\text{min} = 0.0015\pi$ radian/min.

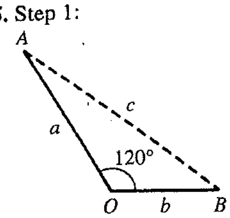
Step 3:

We want to find $-\frac{ds}{dt}$.

Step 4:

$\tan\theta = \frac{80}{s}$ or $s = 80\cot\theta$

35. Step 1:



a = distance from O to A

b = distance from O to B

c = distance from A to B

Step 2:

At the instant in question, $a = 5$ nautical miles, $b = 3$ nautical miles, $\frac{da}{dt} = 14$ knots, and $\frac{db}{dt} = 21$ knots.

Step 3:

We want to find $\frac{dc}{dt}$,

Step 4:

Law of Cosines : $c^2 = a^2 + b^2 - 2ab\cos 120°$

$$c^2 = a^2 + b^2 + ab$$

Step 5:

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}$$

Step 6:

Note that, at the instant in question,

$$c = \sqrt{a^2 + b^2 + ab} = \sqrt{(5)^2 + (3)^2 + (5)(3)} = \sqrt{49} = 7$$

$$2(7)\frac{dc}{dt} = 2(5)(14) + 2(3)(21) + (5)(21) + (3)(14)$$

$$14\frac{dc}{dt} = 413$$

$$\frac{dc}{dt} = 29.5 \text{ knots}$$

The ships are moving apart at a rate of 29.5 knots.