Section 7.3 Exercises

1. In each case, the width of the cross section is $w = 2\sqrt{1-x^2}$.

(a) $A = \pi r^2$, where $r = \dfrac{w}{2}$, so $A(x) = \pi\left(\dfrac{w}{2}\right)^2 = \pi(1-x^2)$.

(b) $A = s^2$, where $s = w$, so $A(x) = w^2 = 4(1-x^2)$.

(c) $A = s^2$, where $s = \dfrac{w}{\sqrt{2}}$, so $A(x) = \left(\dfrac{w}{\sqrt{2}}\right)^2 = 2(1-x^2)$.

(d) $A = \dfrac{\sqrt{3}}{4}w^2$ (see Quick Review Exercise 5), so

$$A(x) = \frac{\sqrt{3}}{4}(2\sqrt{1-x^2})^2 = \sqrt{3}(1-x^2).$$

3. A cross section has width $w = 2\sqrt{x}$ and area

$A(x) = s^2 = \left(\dfrac{w}{\sqrt{2}}\right)^2 = 2x$. The volume is

$$\int_0^4 2x\,dx\left[x^2\right]_0^4 = 16.$$

5. The cross section has width $w = 2\sqrt{1-x^2}$ and area

$A(x) = s^2 = w^2 = 4(1-x^2)$. The volume is

$$\int_{-1}^1 4(1-x^2)\,dx = 4\int_{-1}^1 (1-x^2)\,dx = 4\left[x - \frac{1}{3}x^3\right]_{-1}^1 = \frac{16}{3}.$$

7. The solid is a right circular cone of radius 1 and height 2.

$$V = \frac{1}{3}Bh = \frac{1}{3}(\pi r^2)h = \frac{1}{3}(\pi 1^2)2 = \frac{2}{3}\pi$$

9. A cross section has radius $r = \tan\left(\dfrac{\pi}{4}y\right)$ and area

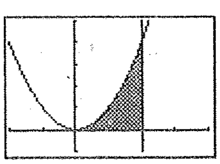
$A(y) = \pi r^2 = \pi\tan^2\left(\dfrac{\pi}{4}y\right)$. The volume is

$$\int_0^1 \pi\tan^2\left(\frac{\pi}{4}y\right)dy = \pi\left[\frac{4}{\pi}\tan\left(\frac{\pi}{4}y\right) - y\right]_0^1$$

$$= \pi\left(\frac{4}{\pi} - 1\right)$$

$$= 4 - \pi.$$

11.



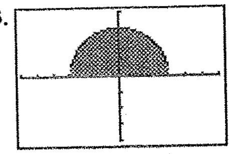
[−2, 4] by [−1, 5]

A cross section has radius $r = x^2$ and area

$A(x) = \pi r^2 = \pi x^4$. The volume is

$$\int_0^2 \pi x^4\,dx = \pi\left[\frac{1}{5}x^5\right]_0^2 = \frac{32\pi}{5}.$$

13.

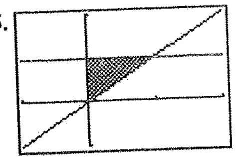


[−6, 6] by [−4, 4]

The solid is a sphere of radius $r = 3$. The volume is

$\dfrac{4}{3}\pi r^3 = 36\pi$.

15.

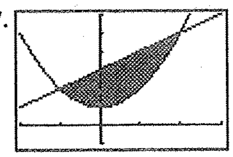


[−1, 2] by [−1, 2]

Use cylindrical shells: A shell has radius y and height y.

The volume is $\displaystyle\int_0^1 2\pi(y)(y)\,dy = 2\pi\left[\frac{1}{3}y^3\right]_0^1 = \frac{2}{3}\pi$.

17.



[−2, 3] by [−1, 6]

The curves intersect when $x^2 + 1 = x + 3$, which is when $x^2 - x - 2 = (x-2)(x+1) = 0$, i.e., when $x = -1$ or $x = 2$. Use washer cross sections: a washer has inner radius $r = x^2 + 1$, outer radius $R = x + 3$, and area

$$A(x) = \pi(R^2 - r^2)$$

$$= \pi\left[(x+3)^2 - (x^2+1)^2\right]$$

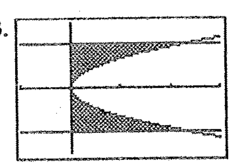
$$= \pi(-x^4 - x^2 + 6x + 8). \text{ The volume is}$$

$$\int_{-1}^2 \pi(-x^4 - x^2 + 6x + 8)\,dx$$

$$= \pi\left[-\frac{1}{5}x^5 - \frac{1}{3}x^3 + 3x^2 + 8x\right]_{-1}^2$$

$$= \pi\left[\left(-\frac{32}{5} - \frac{8}{3} + 12 + 16\right) - \left(\frac{1}{5} + \frac{1}{3} + 3 - 8\right)\right] = \frac{117\pi}{5}.$$

23.



[−1, 3] by [−1.5, 1.5]

A cross section has radius $r = \sqrt{5}y^2$ and area

$A(y) = \pi r^2 = 5\pi y^4$.

The volume is $\displaystyle\int_{-1}^1 5\pi y^4\,dy = \pi\left[y^5\right]_{-1}^1 = 2\pi$.

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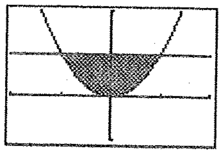
31.



[−2, 2] by [−1, 2]

The curves intersect at $(\pm 1, 1)$.

(a) A cross section has radius $r = 1 - x^2$ and area $A(x) = \pi r^2 = \pi(1-x^2)^2 = \pi(1 - 2x^2 + x^4)$.
The volume is

$$\int_{-1}^{1} \pi(1-2x^2+x^4)\,dx = \pi\left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{1} = \frac{16\pi}{15}.$$

(b) Use cylindrical shells: a shell has radius $2-y$ and height $2\sqrt{y}$. the volume is

$$\int_0^1 2\pi(2-y)(2\sqrt{y})\,dy = 4\pi\int_0^1 (2\sqrt{y} - y^{3/2})\,dy$$

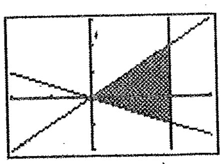
$$= 4\pi\left[\frac{4}{3}y^{3/2} - \frac{2}{5}y^{5/2}\right]_0^1 = \frac{56\pi}{15}.$$

(c) Use cylindrical shells: a shell has radius $y+1$ and height $2\sqrt{y}$. The volume is

$$\int_0^1 2\pi(y+1)(2\sqrt{y})\,dy = 4\pi\int_0^1 (y^{3/2} + \sqrt{y})\,dy$$

$$= 4\pi\left[\frac{2}{5}y^{5/2} + \frac{2}{3}y^{3/2}\right]_0^1 = \frac{64\pi}{15}.$$

35.



[−2, 3] by [−2, 3]

A shell has radius x and height $x - \left(-\frac{x}{2}\right) = \frac{3}{2}x$.

The volume is $\int_0^2 2\pi(x)\left(\frac{3}{2}x\right)dx = \pi\left[x^3\right]_0^2 = 8\pi$.

55. $g'(y) = \frac{dx}{dy} = \frac{1}{2\sqrt{y}}$, and

$$\int_0^2 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy = \int_0^2 \pi\sqrt{4y+1}\,dy$$

$$= \left[\frac{\pi}{6}(4y+1)^{3/2}\right]_0^2$$

$$= \frac{13\pi}{3} \approx 13.614$$

33. A shell has height $12(y^2 - y^3)$.

(a) A shell has radius y. The volume is

$$\int_0^1 2\pi(y)12(y^2-y^3)\,dy = 24\pi\int_0^1 (y^3 - y^4)\,dy$$

$$= 24\pi\left[\frac{1}{4}y^4 - \frac{1}{5}y^5\right]_0^1 = \frac{6\pi}{5}.$$

(b) A shell has radius $1-y$. The volume is

$$\int_0^1 2\pi(1-y)12(y^2-y^3)\,dy$$

$$= 24\pi\int_0^1 (y^4 - 2y^3 + y^2)\,dy$$

$$= 24\pi\left[\frac{1}{5}y^5 - \frac{1}{2}y^4 + \frac{1}{3}y^3\right]_0^1 = \frac{4\pi}{5}.$$

(c) A shell has radius $\frac{8}{5} - y$. The volume is

$$\int_0^1 2\pi\left(\frac{8}{5} - y\right)12(y^2-y^3)dy$$

$$= 24\pi\int_0^1\left(y^4 - \frac{13}{5}y^3 + \frac{8}{5}y^2\right)dy$$

$$= 24\pi\left[\frac{1}{5}y^5 - \frac{13}{20}y^4 + \frac{8}{15}y^3\right]_0^1 = 2\pi.$$

(d) A shell has radius $y + \frac{2}{5}$. The volume is

$$\int_0^1 2\pi\left(y + \frac{2}{5}\right)12\left(y^2 - y^3\right)dy$$

$$= 24\pi\int_0^1\left(-y^4 + \frac{3}{5}y^3 + \frac{2}{5}y^2\right)dx$$

$$= 24\pi\left[-\frac{1}{5}y^5 + \frac{3}{20}y^4 + \frac{2}{15}y^3\right]_0^1 = 2\pi.$$

57. $g'(y) = \frac{dx}{dy} = \frac{1}{2}y^{-1/2}$, and

$$\int_1^3 2\pi\left[y^{1/2} - \left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1+\left[\frac{1}{2}y^{-1/2}\right]^2}\;dy$$

$$= 2\pi\int_1^3\left[y^{1/2} - \left(\frac{1}{3}\right)^{3/2}\right]\sqrt{1+\frac{1}{4y}}\,dy.$$

Using NINT, this evaluates to ≈ 16.110

59. $f'(x) = \frac{dy}{dx} = 2x$, and

$\int_0^2 2\pi r^2\sqrt{1+(2x)^2}\,dx = \int_0^2 2\pi r^2\sqrt{1+4x^2}\,dx$ evaluates, using NINT, to ≈ 53.226.

61. $f'(x) = \frac{dy}{dx} = \frac{1-x}{\sqrt{2x-x^2}}$, and

$$\int_{0.5}^{1.5} 2\pi\sqrt{2x-x^2}\sqrt{1+\left(\frac{1-x}{\sqrt{2x-x^2}}\right)^2}\,dx = 2\pi\int_{0.5}^{1.5} 1\,dx$$

$$= 2\pi\left[x\right]_{0.5}^{1.5}$$

$$= 2\pi \approx 6.283$$