3.1 Derivative of a Function

Definition of Derivative

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

alternate form

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Notation for finding the derivative:

$f'(x)$, y', $\frac{dy}{dx}$, $\frac{d}{dx}$, $\frac{df}{dx}$, $\frac{d}{dx}f(x)$

Ex 1) Find $f'(x)$ if $f(x) = x^2 + 4$ at $x = 1$.

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Ex 2) Find y' for $f(x) = x^2 - 1$ at $a = -2$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Ex 3) Find $\frac{dy}{dx}$ of $f(x) = \sqrt{(x + 2)}$.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Ex 4) Use the definition of the derivative to find f'(1) for $f(x) = \frac{1}{x^2}$.

Ex 5) At what point is the tangent to $f(x) = x^2 + 4x - 1$ horizontal?

problem #29 from2.4

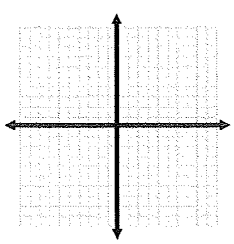
3.2 Differentiability

*Are you able to find a derivative?
*Is there a derivative?
*Can you find the slope at the point?
*Is there a tangent line?

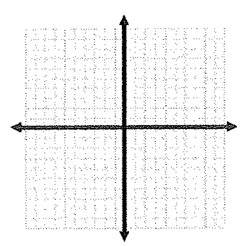
Yes = Differentiable

There are 4 types of non-differentiability:

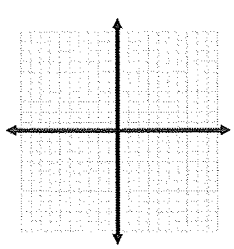
1. Corners



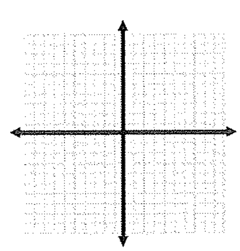
2. Cusp



3. Vertical Tangent



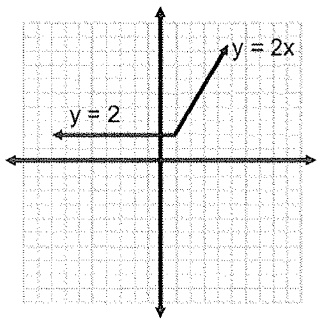
4. Discontinuity



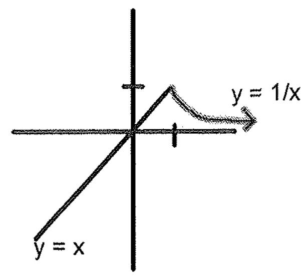
Differentiable

1. Continuous
2. Left-hand and Right-hand derivatives (slopes) must be equal.

Differentiable?



Differentiable?



Differentiability Implies Continuity
If f has a derivative at x = a, then f is continuous at x = a.

3.3 Rules for Differentiation Day 1

Constant rule

$$\frac{d}{dx}(c) = 0$$

Power rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Ex 1) $y = x^3$

Ex 2) $y = x^{10}$

Ex 3) $y = \sqrt{x}$

Ex 4) $y = \frac{1}{x^3}$

Constant multiple rule

$$\frac{d}{dx}\, c \cdot f(x) = c \cdot \frac{d}{dx}\, f(x)$$

Ex 5) $y = 5x^2$

Ex 6) $y = 3x^6$

Ex 7) $y = \dfrac{2}{x^3}$

Ex 8) $f(x) = 6x^3$

A) Find the slope at $x = 2$

B) Where is the slope of the tangent horizontal?

3.3 Rules for Differentiation Day 2

Power Rule

Ex 1) $y = x^4 + 3x^3 - 2x^{-2} + 4\sqrt{x}$

Product Rule

$$\frac{d}{dx}(uv) = uv' + vu'$$

Ex 2) $\frac{d}{dx}(x - 1)(x^2 - 2)$

Quotient Rule

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$$

Ex 3) $\frac{d}{dx}\left(\frac{x^2 + 2x - 1}{x - 4}\right)$

Ex 4) $f(x) = x^4 - x^3 + x^2 - 2x + 6$

Find equations for the tangent and normal lines at $x = 1$.

Ex 5) $f(x) = x^3 + 3x^2 - 3x + 6$

Where is the tangent line horizontal?

Ex 6) Suppose $u(1) = 2$, $u'(1) = 3$, $v(1) = -2$, $v'(1) = 4$

$\frac{d}{dx}(uv)$

$\frac{d}{dx}\, 2u - 4v + 3uv$

$\frac{d}{dx}\frac{u}{v}$

Ex 7) $y = x^4 - x^3 + x^2 - 2x + 6$

$y' =$

$y'' =$

$y''' =$

$y'''' =$

$y''''' =$

39 $y = 2x^3 - 3x^2 - 12x + 20$

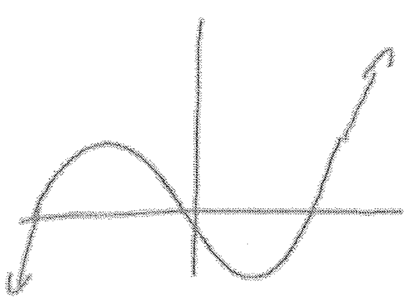
$y' = \boxed{0 = 6x^2 - 6x - 12}$

$0 = 6(x^2 - x - 2)$

$0 = 6(x - 2)(x + 1)$

$x - 2 = 0$ $x + 1 = 0$

$x = 2$ (min) $x = -1$ (max)



3.4 Velocity and Other Rates Day 1

Position, Velocity, Speed, Acceleration

Position = s(t) = position at time t

Δs = displacement

Velocity = v(t)

v(t) = s'(t) = Instantaneous velocity

$$\frac{\Delta s}{\Delta t} = \text{Average velocity}$$

Speed = |v(t)|

Acceleration = a(t) = v'(t) = s''(t)

Ex 1) If $s(t) = t^3 - 3t^2 + 12t + 4$

Find v(3)

Find does v(t) = 0?

Find the speed
at t = 1, 2, 3

Find a(1)

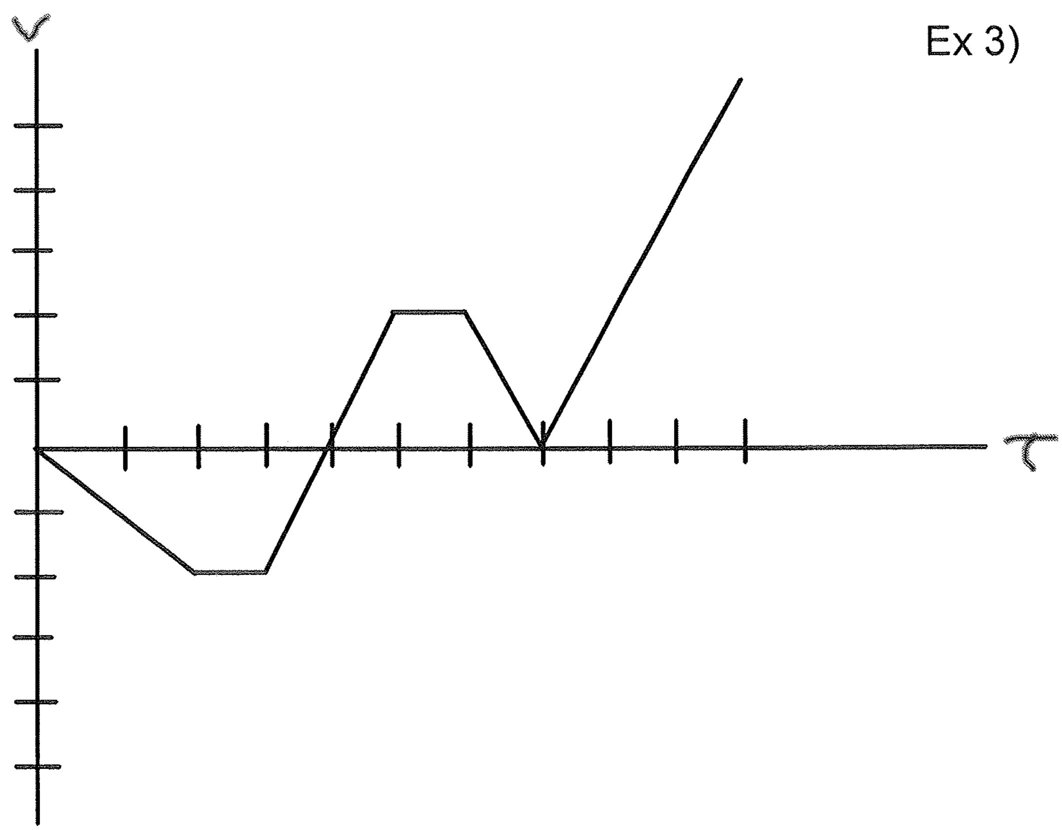
Ex 2) An object is thrown in the air. Its height is modeled by $h(t) = 160t - 16t^2$.

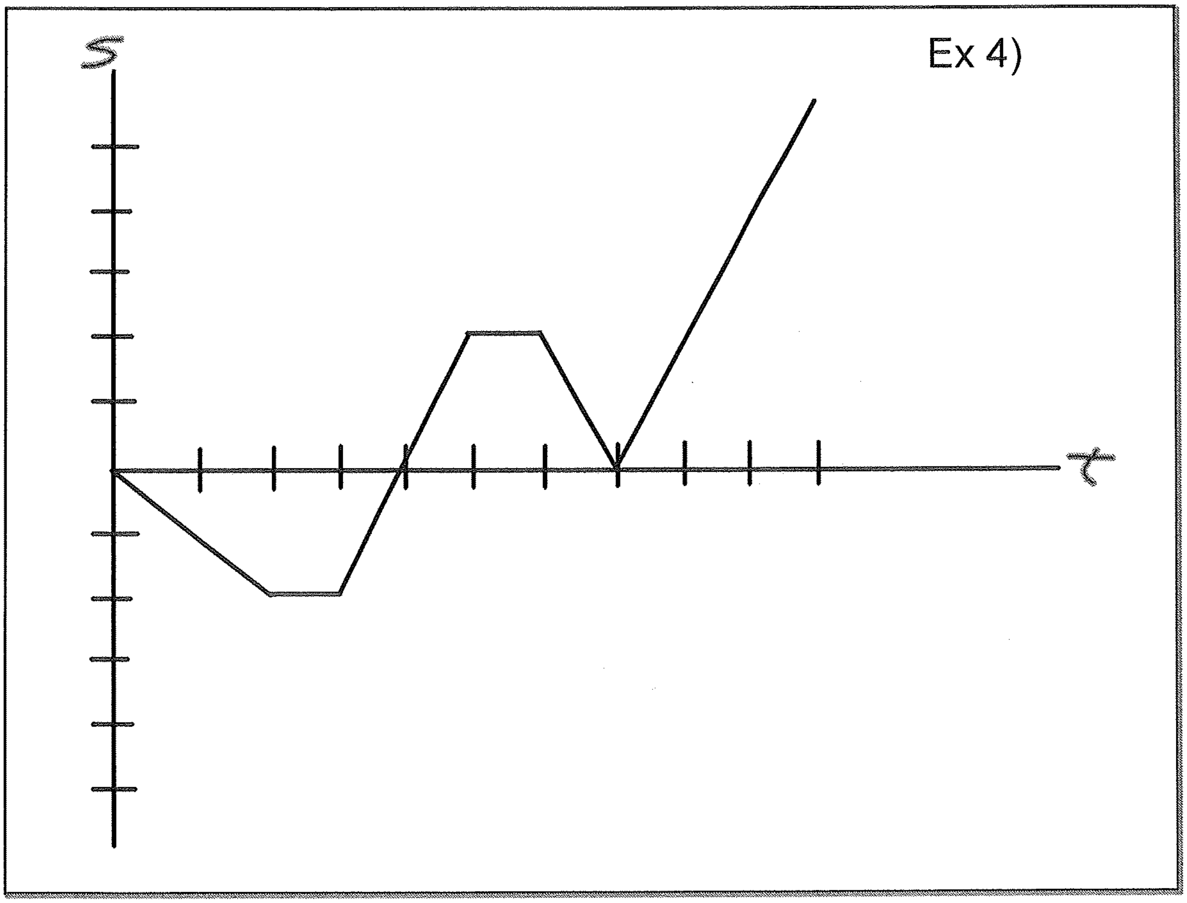
When does it reach its highest point?

How high did it go?

How long was it in the air?

What is the average velocity [0,2] ?





s
Ex 4)
t

3.4 Velocity and Other Rates Day 2

What does the derivative mean?

- Slope
- How fast something is changing
- Instantaneous rate of change

Ex 1) Write an equation relating surface area of a cube with its side length

Find the instantaneous rate of change for surface area with respect to s.

Evaluate A' (1) and A' (2)

Ex 2) Write an equation relating surface area of a sphere with its radius.

Find the instantaneous rate of change for surface area with respect to r.

Evaluate A' (1) and A' (2)

Ex 4) A bullet fired straight up from the moon's surface would reach a height of $s = 832t - 2.6t^2$ after t seconds. How long would it take the bullet to to get back down?

Ex 5) A particle moves along a line so that its position at time t is given by $s(t) = t^3 - 6t^2 + 8t + 2$ where s is measured in meters and t is measured in seconds with $t \geq 0$.

a) Find displacement during first 5 sec.

b) Find average velocity during first 5 sec.

c) When does the particle change direction.

d) Where is the particle when s is a minimum?

Ex 6) A body's velocity at time t sec is

$$v = 2t^3 - 9t^2 + 12t - 5.$$

Find the body's speed each time the acceleration is zero

3.4 Velocity and Other Rates Day 2

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3.5 Derivatives of Trigonometric Functions

MEMORIZE

$y = \sin x$	$y' = \cos x$
$y = \cos x$	$y' = -\sin x$
$y = \tan x$	$y' = \sec^2 x$
$y = \sec x$	$y' = \sec x \tan x$
$y = \csc x$	$y' = -\csc x \cot x$
$y = \cot x$	$y' = -\csc^2 x$

Find y'

Ex 1) $y = \sin x - \tan x + 5x$

Ex 2) $y = x \csc x$

Find y'

Ex 3) $y = \dfrac{\sin x + \cos x}{\cos x}$

Find y'

Ex 4) $y = \dfrac{\cos x}{1 + \sin x}$

Ex 5) Write an equation for the tangent line and normal line to graph $y = x + \cos x$ at $(0,1)$

23 $\quad s = t^3 - 6t^2 + 9t$

$a(t) = ? \qquad v(t) = 0$

$v = 3t^2 - 12t + 9$

$a = 6t - 12 \rightarrow a(1) = 6 \cdot 1 - 12 = -6$

$a(3) = 6 \cdot 3 - 12 = 6$

$3t^2 - 12t + 9 = 0$

$3(t^2 - 4t + 3) = 0$

$3(t-1)(t-3) = 0$

$t = 1 \quad t = 3$

3.5 Derivatives of Trigonometric Functions Day 2

$y = \sin x$	$y' = \cos x$
$y = \cos x$	$y' = -\sin x$
$y = \tan x$	$y' = \sec^2 x$
$y = \sec x$	$y' = \sec x \tan x$
$y = \csc x$	$y' = -\csc x \cot x$
$y = \cot x$	$y' = -\csc^2 x$

Find the derivative of each.

Ex 1) $y = \sec x$

Ex 2) $y = \csc x$

Ex 3) Which of the following is an equation of the normal line to $y = \sin x + \cos x$ at $x = \pi$?

A. $y = -x + \pi - 1$
B. $y = x - \pi - 1$
C. $y = x - \pi + 1$
D. $y = x + \pi + 1$
E. $y = x + \pi - 1$

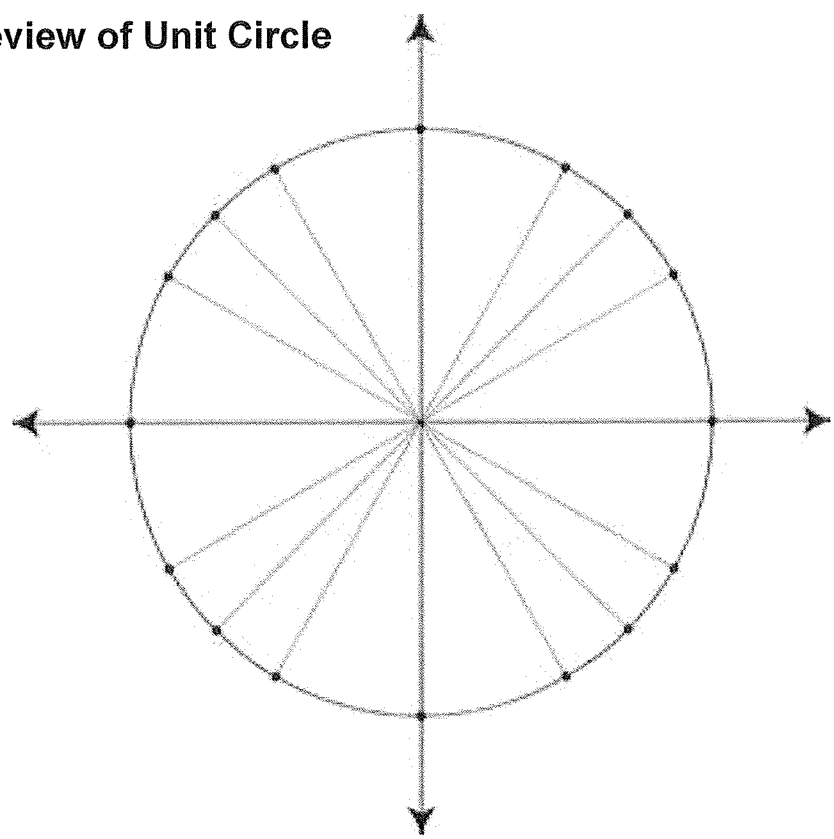
Ex 4) Find y'' if $y = x \sin x$

A. $-x \sin x$
B. $x \cos x + \sin x$
C. $-x \sin x + 2 \cos x$
D. $x \sin x$
E. $-\sin x + \cos x$

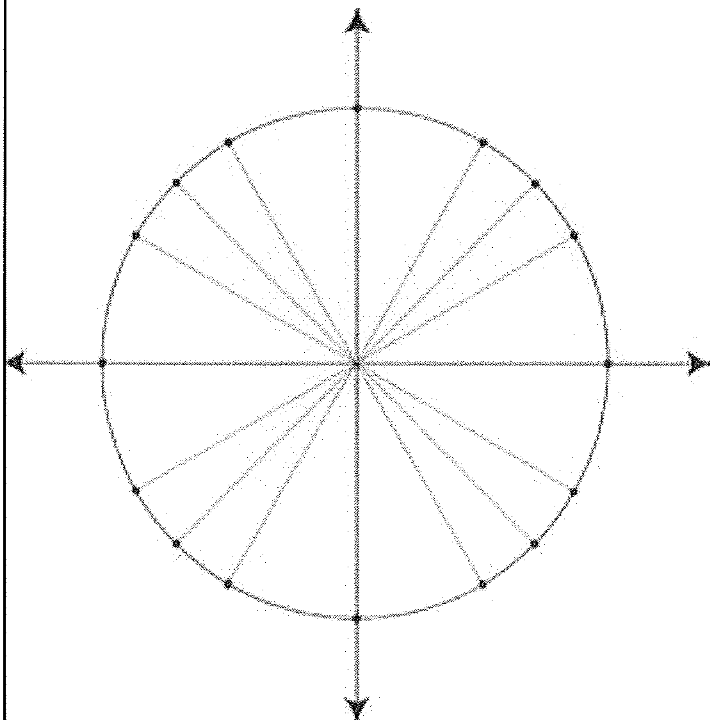
Ex 5) A body is moving in simple harmonic motion with position s = 3 + sin t. At which of the following times is the velocity zero?

A. $t = 0$
B. $t = \pi/4$
C. $t = \pi/2$
D. $t = \pi$
E. none of these

Ex 6) Review of Unit Circle



Pythagorean Identities



Ex 9)

$\csc \frac{11\pi}{6}$

$\cot \frac{7\pi}{4}$

$\tan \frac{5\pi}{4}$

$\sec \frac{7\pi}{6}$

3.6 Chain Rule

Order of Operations

Ex 1) $y = (3x + 1)^2$

Evaluate at $x = 1$

Ex 2) $y = \cos(2x + \pi)$

Evaluate at $x = \pi/2$

Ex 3) $y = (3x + 1)^2$

Find $\frac{dy}{dx}$

Ex 4) $y = (3x^2 + 6x)^5$

Find $\frac{dy}{dx}$

Ex 5) $y = (3x + 1)^2$

Find $\frac{dy}{dx}$

Ex 6) $y = (3x^2 + 6x)^5$

Find $\frac{dy}{dx}$

Ex 7) $y = (x^2 + 2x + 3)^3$

Find $\frac{dy}{dx}$

Ex 8) $y = \tan(5x)$

Find $\frac{dy}{dx}$

Ex 9) $y = \sin(x^2 + 4)$

Find $\frac{dy}{dx}$

Ex 10) $y = \cos(2x + 3)^3$

Find $\frac{dy}{dx}$

Ex 11) $y = 2\sin\sqrt{x^2 - 9}$

Find $\frac{dy}{dx}$

Ex 12) $y = \sin^3 x \cdot \tan(4x)$

Find $\frac{dy}{dx}$

Ex 13) $y = \frac{x}{\sqrt{1 + x^2}}$

Find $\frac{dy}{dx}$

Ex 14) $y = (1 + \cos^2(3x))^5$

Find $\frac{dy}{dx}$

3.6 The Chain Rule Day 2

Ex 1) Let $f(u) = u^3 + u$ Find $(f \circ g)'$

$u = g(x) = 4x$

Ex 2) Given

$f(1) = 2$ $f'(1) = 3$ $f'(2) = -4$

$g(1) = 2$ $g'(1) = -3$ $g'(2) = 5$

If $h(x) = f(g(x))$

Find $h'(1)$

Ex 3) $x = 3\cos(2t)$

$y = 2\sin(3t)$

Find $\frac{dy}{dx}$ $t = \pi/3$

Ex 4) $x = 3t^2 + 2$

$y = t^3$

Find $\frac{dy}{dx}$ $t = 1$

1. $\frac{d}{dx} \sin^2(x^3)$

2. $f(x) = \sec(2x)$. Find $f'(\pi/6)$

3. Write an equation for the tangent to the graph of $y = x(1 - 2x)^2$ at (1, 1)

A. $y=2x+1$ B. $y= -4x+5$

C. $y = -2x - 2$ D. $y = 5x-4$

4. $y = (1 + \cos^2(7x))^3$

3.7 Implicit Differentiation

1. Differentiate both sides with respect to x.
2. Get all terms with dy/dx to one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx

Ex 1) $y = x^2$ Find $\frac{dy}{dx}$

Ex 2) $x = y^2$ Find $\frac{dy}{dx}$

Ex 3) $2x^3 + 5y^2 = 10$

Find $\frac{dy}{dx}$

Ex 4) $x^5 + 4y^3 - 2y^2 = 50$

Find $\frac{dy}{dx}$

Ex 5) $x^5 + 4xy^3 - 5y^5 = 4$

Find $\frac{dy}{dx}$

3.7 Implicit Differentiation Day 2

1. Differentiate both sides with respect to x.
2. Get all terms with dy/dx to one side of the equation.
3. Factor out dy/dx.
4. Solve for dy/dx

Ex 1) Find the slope of the tangent to $y^2 - x^2 = 1$ at $(1, \sqrt{2})$

Ex 2) $x^2 - xy + y^2 = 1$

Find $\frac{dy}{dx}$

Find $\frac{d^2y}{dx^2}$ ← Do on next slide

$\frac{dy}{dx} = y' =$

Now find $\frac{d^2y}{dx^2} = y''$

Ex 3) $y^2 + 2x - 4y - 1 = 0$ Find the tangent and normal line at (-2, 1).

Ex 4) $x \sin 2y = y \cos 2x$

Find the tangent and normal line at $(\pi/4, \pi/2)$

Ex 5) $x^2 \cos^2 y - \sin y = 0$

Find the tangent and normal line at $(0, \pi)$

3.8 Derivatives of Inverse Trigonometric Functions
Day 1

What is an inverse?

$\sin^{-1}x = \arcsin x$

$\cos^{-1}x = \arccos x$

$\tan^{-1}x = \arctan x$

Graphs

$y = \sin x$	$y = \sin^{-1}x$

$\sin x = 0.5$

MEMORIZE

$$\frac{d}{dx} \sin^{-1}x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \cos^{-1}x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \tan^{-1}x = \frac{1}{1 + x^2}$$

Ex 1)

$$\frac{d}{dx} \tan^{-1}(x^2) =$$

Ex 2)

$$\frac{d}{dx} \tan^{-1}\sqrt{(3x)} =$$

Ex 3)

$$\frac{d}{dx} \sin^{-1}\frac{x}{3} =$$

Ex 4)

$$\frac{d}{dx} \arcsin(2x^7 + 1) =$$

Ex 5)

$$\frac{d}{dx} \; x\cos^{-1}x + \sqrt{(1 - x^2)} =$$

3.8 Derivatives of Inverse Trigonometric Functions

Day 2

MEMORIZE

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\csc^{-1}x = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\frac{d}{dx}\cot^{-1}x = -\frac{1}{1 + x^2}$$

Ex 1)

$$\frac{d}{dx}\sec^{-1}(x^3) =$$

Ex 2)

$$\frac{d}{dx}\cot^{-1}(3x) =$$

Ex 3)

$$\frac{d}{dx} \csc^{-1} \frac{x}{3} =$$

Ex 4)

$$\frac{d}{dx} \cot^{-1} \sqrt{(x)} =$$

Ex 5)

$$\frac{d}{dx} \sec^{-1} x + \sqrt{(x^2 + 1)} =$$

Ex 6) Write an equation for the line tangent to $y = \tan^{-1} x$ at $x = 1$

Ex 7) Write an equation for the line tangent to $y = \arcsin x$ at $x = 0.5$

Ex 8) $y = 3x^2 + 4x + 2$

$y(1) =$

$y'(1) =$

$y^{-1}(9) =$

$(y^{-1})'(9) =$

Ex 9)

If $f(x) = 3x^2 - x$ and $g(x) = f^{-1}(x)$, then $g'(10) =$

3.9 Derivatives of Exponential and Logarithmic Functions
Day 1

$$\frac{d}{dx} a^u = a^u \ln a \frac{du}{dx}$$

$$\frac{d}{dx} \log_a u = \frac{1}{u \ln a} \frac{du}{dx}$$

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$\frac{d}{dx} \ln u = \frac{1}{u} \frac{du}{dx}$$

Ex 1) $y = e^{2x}$

$y' =$

Ex 2) $y = e^{-5x}$

$y' =$

Ex 3) $y = xe^{2x} + 2x^5$

$y' =$

Ex 4) $y = x^2e^{(x^3)}$

$y' =$

Ex 5) $y = 9^{-x}$

$y' =$

Ex 6) $y = 3^{\cos x}$

$y' =$

Ex 7) $y = xe^2 - e^x$

$y' =$

Ex 8) $y = \ln x^2$

$y' =$

Ex 9) $y = (\ln x)^2$

$y' =$

Ex 10) $y = \ln \dfrac{10}{x}$

$y' =$

Ex 11) $y = x \ln x - x$

$y' =$

Ex 12) $y = \log_5 \sqrt{x}$	Ex 13) $y = \log_3(1 + x \ln 3)$
$y' =$	$y' =$

3.9 Derivatives of Inverse Trigonometric Functions Day 2

Ex 1) $y = e^{-x/4}$

$y' =$

Ex 2) $y = e^{\sqrt{x}}$

$y' =$

Ex 3) $y = \ln(\ln x^2)$

$y' =$

Ex 4) At what point on the graph of $y = 2e^x - 1$ is the tangent line perpendicular to the line $y = -3x + 2$?

Ex 5) A line with slope m passes through the origin and is tangent to y = ln (x/3). What is the value of m?

Ex 6) The spread of flu in a certain school is modeled by the equation

$$P(t) = \frac{200}{1 + e^{5-t}}$$

P = Population
t = days

Estimate the initial number of students with the flu.

How fast is it spreading after 4 days?

Ex 7) Which of the following give the slope of the tangent line to the graph of $y = 2^{1-x}$ at $x = 2$?

a. -1/2
b. 1/2
c. -2
d. 2
e. -(ln 2)/2