Sem 1 Final Exam

1. Let f be the function defined by $f(x) = k\sqrt{x} - \ln x$ for $x > 0$, where k is a positive constant.

(a) Find $f'(x)$ and $f''(x)$.

$$f' \quad \frac{1}{2}kx^{-1/2} - \frac{1}{x} = \frac{k}{2\sqrt{x}} - \frac{1}{x} \quad +2$$

$$f'' = -\frac{1}{4}kx^{-3/2} - -\frac{1}{x^2} = -\frac{k}{4x^{3/2}} + \frac{1}{x^2} \quad +2$$

(b) For what value of the constant k does f have a critical point at $x = 1$? For this value of k, determine whether f has a relative minimum, relative maximum, or neither at $x = 1$. Justify your answer.

+1

$$0 = \frac{k}{2\sqrt{1}} - \frac{1}{1}$$

$$1 = \frac{k}{2} \qquad k = 2 \quad +1$$

$$f' = \frac{1}{\sqrt{x}} - \frac{1}{x}$$

f': $-$ | 1 | $+$

+1

minimum

(c) For a certain value of the constant k, the graph of f has a point of inflection on the x-axis. Find this value of k.

$\boxed{x = e^4}$

$$0 = \frac{-k}{4x^{3/2}} + \frac{1}{x^2} \qquad \frac{k}{4x^{3/2}} = \frac{1}{x^2} \qquad \boxed{k = \frac{4}{e^2}} = .541 \quad +2$$

$$\frac{k}{4} = \frac{x^{3/2}}{x^2} \qquad \frac{k}{4} = \frac{1}{\sqrt{x}}$$

$$0 = k\sqrt{x} - \ln x \qquad \boxed{k = \frac{\ln x}{\sqrt{x}}} \qquad \boxed{k = \frac{4}{\sqrt{x}}}$$

$$\frac{k\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}} = \frac{\ln x}{\sqrt{x}} \qquad \ln x = 4$$

e e

2. A cubic polynomial function f is defined by $f(x) = 4x^3 + ax^2 + bx + k$, where a, b, and k are constants. The function f has a local minimum at $x = -1$, and the graph of f has a point of inflection at $x = -2$

(a) Find the values of a and b.

$f' = 12x^2 + 2ax + b \rightarrow 12x^2 + 48x + b = 0$

$12(-1)^2 + 48(-1) + b = 0$

$12 - 48 + b = 0 \quad \boxed{b = 36}$

$f'' = 24x + 2a \qquad 24x + 2a = 0$

$24(-2) + 2a = 0$

$2a = 48 \quad \boxed{a = 24}$

(b) Find any other relative extrema for f.

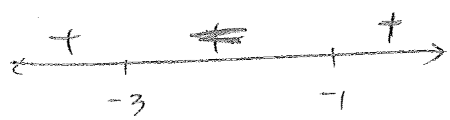
$f(x) = 4x^3 + 24x^2 + 36x$

$f' = 12x^2 + 48x + 36 = 0$

$12(x^2 + 4x + 3) = 0$

$12(x+1)(x+3) = 0$

$x = -1 \qquad x = -3$



(c) Find any y–intercept(s).

$f(0) = 4 \cdot 0 + a \cdot 0 + b \cdot 0 + k$

$f(0) = k = 1$

(d) Write an equation for the line tangent to the graph at $x = -1$.

$f(-1) = -4 + 24 - 36 + 1 = -15$

$y + 15 = 0(x+1)$

$y = -15$

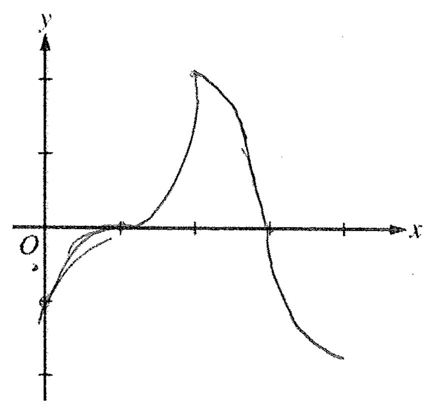
x	0	$0 < x < 1$	1	$1 < x < 2$	2	$2 < x < 3$	3	$3 < x < 4$
$f(x)$	-1	Negative	0	Positive	2	Positive	0	Negative
$f'(x)$	4	Positive	0	Positive	DNE	Negative	-3	Negative
$f''(x)$	-2	Negative	0	Positive	DNE	Negative	0	Positive

3. Let f be a function that is continuous on the interval $[0, 4)$. The function f is twice differentiable except at $x = 2$. The function and its derivatives have the properties indicated in the table above, where DNE indicates that the derivatives of f do not exixt at $x = 2$.

(a) For $0 < x < 4$, find all values of x at which f has a relative extremum. Determine whether f has a relative maximum or a relative minimum at each of these values. Justify your answer.

$X = 2$ is a max

(b) On the axes, sketch the graph of a function that has all the characteristics of f.



(c) For the function f, find all values of x for $0 < x < 4$, at which the graph of f has a point of inflection. Justify your answer.

$X = 1$ $x = 2$ $x = 3$

(d) Write an equation for the line tangent to f at $x = 0$

$y + 1 = 4(x - 0)$ $y + 1 = 4x$

$\boxed{y = 4x - 1}$

4. A particle moves along the x-axis in such a way that its position at time t is given by $x = 3t^4 - 16t^3 + 24t^2$ for $-5 \le t \le 5$.

(a) Determine the velocity and acceleration of the particle at time t.

$V = 12t^3 - 48t^2 + 48t$

$a = 36t^2 - 96t + 48$

(b) At what values of t is the particle at rest?

$0 = 12t^3 - 48t^2 + 48t$

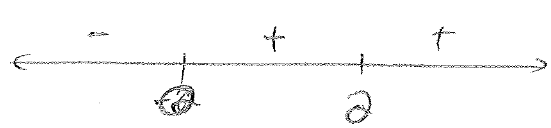
$= 12t(t^2 - 4t + 4)$

$= 12t(t - 2)(t - 2)$

$t = 0$ $\quad$ $t = 2$

(c) At what values of t does the particle change direction?

$V =$



$t = 0$

(d) What is the velocity when the acceleration is first zero?

$0 = 36t^2 - 96t + 48$

$= 12(3t^2 - 8t + 4)$

$= 12(3t - 2)(t - 2)$

$3t - 2 = 0$ $\quad$ $t = 2$ $\rightarrow$ [illegible]

$3t = 2$

$t = \frac{2}{3}$

$V = 12\left(\frac{2}{3}\right)^3 - 48\left(\frac{2}{3}\right)^2 + 48\left(\frac{2}{3}\right)$

5. Given the curve $x + xy + 2y^2 = 6$.

(a) Find an expression for the slope of the curve at any point (x,y) on the curve.

$$1 + x\frac{dy}{dx} + y + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(x+4y) = -y-1$$

$$\boxed{\frac{dy}{dx} = \frac{-y-1}{x+4y}}$$

(b) Write an equation for the line tangent to the curve at the point (2, 1).

$$\boxed{(y-1) = \frac{-1}{3}(x-2)}$$

$$m = \frac{-1-1}{2+4\cdot 1} = \frac{-2}{6}$$

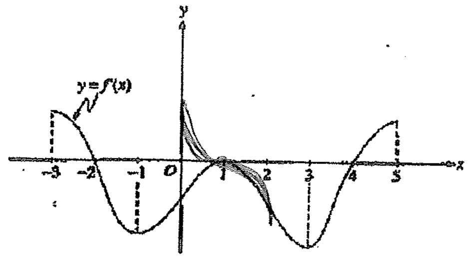
$$\boxed{y = \frac{-1}{3}x + \frac{5}{3}}$$

(c) Find the coordinates of all other points on the curve with slope equal to the slope at (2, 1).

$$\frac{-1}{3} = \frac{-y-1}{x+4y}$$

$$-x - 4y = -3y - 3$$
$$+4y \quad +4y$$
$$-x = y - 3$$
$$y = 3 - x$$

$$(x, 3-x)$$
$$(3-y, y)$$



Note: This is the graph of the derivative of f, not the graph of f.

6. The figure above shows the graph of $f'(x)$, the derivative of function f. The domain of f is the set of all real numbers x such that $-3 < x < 5$.

3 (a) For what value(s) of x does f have a horizontal tangent line? Justify your answer.

$x = -2$, $x = 1$, $x = 4$ $\rightarrow$ $f' = 0$

2 (b) For what value(s) of x does f have a relative maximum? Justify your answer.

$x = -2$ f' switches from + to −

2 (c) On what intervals is the graph of f concave upward? Justify your answer.

$-1 < x < 1$ $3 < x < 5$

the slope of f' is positive

2 (d) Suppose that $f(1) = 0$. In the same xy-plane provided above, draw a sketch that shows the general shape of the graph of the function f on the open interval $0 < x < 2$.