11-5 day 2

I. Use the limit process to find the area of the region between the graph of the function and the x-axis over the given interval.

Ex 1) $f(x) = 3x - x^2$ $\left[1, 2\right]$ $a=1$ $b=2$

$$\lim_{n\to\infty}\sum_{i=1}^{n} f\left(1+\frac{(2-1)i}{n}\right)\left(\frac{2-1}{n}\right) = \sum_{i=1}^{n} f\left(1+\frac{i}{n}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left[3\left(1+\frac{i}{n}\right)-\left(1+\frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[3+\frac{3i}{n}-\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left(2+\frac{i}{n}-\frac{i^2}{n^2}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{2}{n}+\frac{i}{n^2}-\frac{i^2}{n^3}$$

$$= \frac{1}{n}\sum_{i=1}^{n} 2 + \frac{1}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^3}\sum_{i=1}^{n} i^2 = \frac{1}{n}\cdot 2n + \frac{1}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 2 + \frac{n+1}{2n} - \left(\frac{2n^2+3n+1}{6n^2}\right) =$$

$$= \lim_{n\to\infty} 2 + \lim_{n\to\infty}\frac{n+1}{2n} - \lim_{n\to\infty}\frac{2n^2+3n+1}{6n^2} = 2+\frac{1}{2}-\frac{2}{6} = \boxed{\frac{13}{6}}$$

Ex 2) $f(x) = 4x + 1 \quad [0, 1]$

$a = 0 \quad b = 1$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(0 + \frac{(1-0)i}{n}\right) \cdot \left(\frac{1-0}{n}\right) = \sum_{i=1}^{n} f\left(\frac{1i}{n}\right) \cdot \left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n} \left[4\left(\frac{i}{n}\right) + 1\right] \cdot \left(\frac{1}{n}\right) = \sum_{i=1}^{n} \frac{4i}{n^2} + \frac{1}{n}$$

$$= \frac{4}{n^2} \sum_{i=1}^{n} i + \frac{1}{n} \sum_{i=1}^{n} 1$$

$$= \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n} \cdot n$$

$$= \frac{2(n+1)}{n} + \frac{n}{n} = \frac{3n+2}{n}$$

$$\lim_{n\to\infty} \frac{3n+2}{n} = \boxed{3}$$

Ex 3) $f(x) = 2 - x^2$ $\quad [-1,\ 1]$ $\quad a = -1 \quad b = 1$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\left(-1+\frac{(1--1)i}{n}\right)\cdot\left(\frac{1--1}{n}\right) = \sum_{i=1}^{n} f\left(-1+\frac{2i}{n}\right)\cdot\left(\frac{2}{n}\right)$$

$$\sum_{i=1}^{n} 2-\left[\left(-1+\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n} 2-\left[1-2\left(\frac{2i}{n}\right)+\frac{4i^2}{n^2}\right]\left(\frac{2}{n}\right)$$

$$\sum_{i=1}^{n} 2-\left[1-\frac{4i}{n}+\frac{4i^2}{n^2}\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left(1+\frac{4i}{n}-\frac{4i^2}{n^2}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n} \frac{2}{n}+\frac{8i}{n^2}-\frac{8i^2}{n^3} = \frac{2}{n}\sum_{i=1}^{n} 1 + \frac{8}{n^2}\sum_{i=1}^{n} i - \frac{8}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{2}{n}\cdot n + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) - \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= 2 + \frac{4(n+1)}{n} - \frac{4(2n^2+3n+1)}{3n^2} = 2 + \frac{4n+4}{n} - \frac{8n^2+12n+4}{3n^2}$$

$$= \lim_{n\to\infty} 2 + \frac{4n+4}{n} - \frac{8n^2+12n+4}{3n^2} = 2+4-\frac{8}{3} = \boxed{\frac{10}{3}}$$